Anti-self-dual instantons with Lagrangian boundary conditions II: Bubbling

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Abstract

We study bubbling phenomena of anti-self-dual instantons on $\mathbb{H}^2 \times \Sigma$, where Σ is a closed Riemann surface. The restriction of the instanton to each boundary slice $\{z\} \times \Sigma$, $z \in \partial\mathbb{H}^2$ is required to lie in a Lagrangian submanifold of the moduli space of flat connections over Σ that arises from the restrictions to the boundary of flat connections on a handle body.

We establish an energy quantization result for sequences of instantons with bounded energy near $\{0\} \times \Sigma$: Either their curvature is in fact uniformly bounded in a neighbourhood of that slice (leading to a compactness result) or there is a concentration of some minimum quantum of energy. We moreover obtain a removable singularity result for instantons with finite energy in a punctured neighbourhood of $\{0\} \times \Sigma$. This completes the analytic foundations for the construction of an instanton Floer homology for 3-manifolds with boundary. This Floer homology is an intermediate object in the program proposed by Salamon for the proof of the Atiyah-Floer conjecture for homology-3-spheres.

In the interior case, for anti-self-instantons on $\mathbb{R}^2 \times \Sigma$, our methods provide a new approach to the removable singularity theorem by Sibner-Sibner for codimension 2 singularities with a holonomy condition.

1 Introduction

The aim of this paper is to complete the analytic foundations for the definition of instanton Floer homology groups $\mathrm{HF}_*^{\mathrm{inst}}(M, L_Y)$ begun in [W3]. Here M is a compact, oriented 3-manifold with boundary $\partial M = \Sigma$ and $L_Y \subset M_\Sigma$ is a (singular) Lagrangian submanifold of the moduli space M_Σ of flat connections on the trivial SU(2)-bundle over Σ. This Lagrangian is $L_Y := \mathcal{L}_Y / \mathcal{G}^{1,p}(\Sigma)$, where $\mathcal{L}_Y \subset \mathcal{A}^{0,p}(\Sigma)$ is a Lagrangian submanifold of the space of L^p-connections given by the L^p-closure of the flat connections on a handle body Y restricted to $\partial Y = \Sigma$. Here $\mathcal{G}^{1,p}(\Sigma)$ is the group of $W^{1,p}$-gauge transformations, and L_Y is actually independent of the choice of $p > 2$.

This Floer homology serves as intermediate object in the program proposed by Salamon [Sa] for the proof of the Atiyah-Floer conjecture for homology-3-spheres.

Fukaya [Fu] was the first to suggest the use of Lagrangian boundary conditions in order to define a Floer homology for 3-manifolds with boundary. His setup uses nontrivial bundles (where the moduli spaces of flat connections are smooth manifolds) and thus cannot immediately be used in the context of the Atiyah-Floer conjecture, where the bundles are necessarily trivial and thus the moduli spaces of flat connections are singular. Our approach is to define $\mathrm{HF}^*_{\mathrm{inst}}(M, L_Y)$ from the moduli spaces of anti-self-dual instantons on $\mathbb{R} \times M$ with Lagrangian boundary condition in L_Y, i.e. from the gauge equivalence classes of connections $\Xi \in \mathcal{A}(\mathbb{R} \times M)$ satisfying the boundary value problem

$$\left\{ \begin{array}{l} F_\Xi + *F_\Xi = 0, \\ \Xi|_{\{s\}\times\partial M} \in \mathcal{L}_Y \quad \forall s \in \mathbb{R}. \end{array} \right. \tag{1}$$

Note that the boundary condition is nonlocal: It firstly asserts the local condition that the connection is flat on each boundary slice; but secondly its holonomy has to vanish around those loops in Σ that are contractible in Y, which is a global condition.

In [W3] we describe this approach in full detail and we establish the elliptic theory for this boundary value problem (allowing for a larger class of Lagrangian boundary conditions). Fix $p > 2$, then every $W^{1,p}_{\mathrm{loc}}$-solution is gauge equivalent to a smooth solution and the following analogue of Uhlenbeck compactness is true: Every sequence of solutions with locally L^p-bounded curvature is gauge equivalent to a sequence that contains a $\mathcal{C}^\infty$-convergent subsequence.

In this paper we address the question of bubbling: What happens if a sequence of solutions has bounded energy $\int_{\mathbb{R}\times M} |F_\Xi|^2 < \infty$ but its curvature F_Ξ is not locally L^p-bounded for any $p > 2$?

In the case of a 4-manifold without boundary this question is answered by the compactification of the moduli space of anti-self-dual instantons leading to the Donaldson invariants of smooth 4-manifolds [D] and to the instanton Floer homology groups of closed 3-manifolds [Fl]. This compactification is described in terms of trees of anti-self-dual instantons on S^4 that 'bubble off' at isolated points on the original 4-manifold. In the case of the present boundary value problem, we do not attempt this compactification but only establish the relevant facts for the definition of the Floer homology groups. So the answer comes in two parts (that will be stated more precisely in theorems 1.2 and 1.5):

Energy quantization: If the curvature is not uniformly bounded near an interior point $x \in \mathbb{R} \times \mathrm{int}\, M$ or near a boundary slice $\{s\} \times \Sigma \subset \mathbb{R} \times \partial M$, then there is a minimum energy $\varepsilon_0 > 0$ that concentrates at this point or slice.

Removal of singularities: Every smooth finite energy solution on the complement of an interior point or a boundary slice can be put into a gauge in which it extends to a solution over the full manifold.

In the case of interior points, these are the two wellknown analytic ingredients for the compactification of the moduli space (see e.g. [U1] for Uhlenbeck's removable singularity theorem). The anti-self-dual instantons on S^4 are obtained by rescaling the connections near the bubbling point x. The limit object then is an instanton on $\mathbb{R}^4$ whose singularity at infinity can be removed resulting in an instanton on a nontrivial bundle over S^4.

In the case of bubbling at the boundary, one might also find instantons on S^4 bubbling off at boundary points. These would arise from sequences of solutions Ξ^ν and interior points x^ν with distance $t^\nu \to 0$ to the boundary $\mathbb{R} \times \partial M$, where the curvature $|F_{\Xi^\nu}(x^\nu)| = (R^\nu)^2$ blows up at a rate such that $R^\nu t^\nu \to \infty$. If $R^\nu t^\nu$ stays bounded, then the standard rescaling construction will lead to anti-self-dual instantons on increasingly large domains of the half space. In [Sa] it was conjectured that there is an energy quantization for the limit objects – anti-self-dual instantons on the half space. However, the local rescaling construction looses the global part of the boundary condition. With only the slicewise flatness as boundary condition, one cannot expect to obtain better convergence than weak $W^{1,p}$-convergence (for any $p < \infty$) up to the boundary. In the interior, one of course has smooth convergence, and thus might find a nontrivial limit object. However, in case $R^\nu t^\nu \to 0$, even the limit object might be trivial if the blowup is in the curvature part for which one does not have $\mathcal{C}^0$-convergence up to the boundary. [1]

This discussion suggests a more global analysis of the bubbling phenomenon taking into account the full Σ-slices and localizing only in the two other variables. An adapted rescaling construction seems to lead to holomorphic discs in the space of connections over Σ (with the Hodge operator as complex structure) with Lagrangian boundary conditions. We do not have a precise convergence statement. However, we were able to prove the corresponding energy quantization result by purely analytic means – after all using partial convergence results for the naive local rescaling construction described above.

Before giving the precise statements of our main results we introduce the setup and some basic notation. (For more details on gauge theory and the notation used here see [W2] or [W1].) Throughout this paper, we are working in a small neighbourhood of a boundary slice of a Riemannian 4-manifold with a boundary space-time splitting in the sense of [W3, Def 1.2]. So we are considering the following local model.

We denote by $B_r(x_0) \subset \mathbb{R}^n$ the closed ball of radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$. The intersection of a ball with the half space

$$\mathbb{H}^n := \{(s_1, \dots, s_{n-1}, t) \in \mathbb{R}^n \mid t \geq 0\}$$

[1] Writing $\Xi = \Phi \mathrm{d}s + \Psi \mathrm{d}t + A$ near the boundary $\{t = 0\}$ and assuming $p > 4$, one obtains $W^{2,p}$-bounds for Ξ except for the second ∂_s, ∂_t-derivatives of the connections $A(s,t)$ on the Σ-slices. These bounds suffice to obtain $\mathcal{C}^0$-convergence for the curvature component F_A, but not for $B_s = \partial_s A + \mathrm{d}_A \Phi$. The latter requires full $W^{2,p}$-bounds, which would only result from a Lagrangian boundary condition coupled with the Cauchy-Riemann equation for A as a function with values in $\mathcal{A}^p(\Sigma)$, c.f. [W3].

is denoted by

$$D_r(x_0) := B_r(x_0) \cap \mathbb{H}^n.$$

Moreover, we write $D := D_{r_0}(0) \subset \mathbb{H}^2$ for the 2-dimensional half ball centered at 0 of some fixed radius r_0. Next, let Σ be a closed Riemann surface. Now the local model is the trivial SU(2)-bundle over the Riemannian 4-manifold

$$(\, D \times \Sigma\, ,\; \mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}\,).$$

Here $g_{s,t}$ is a family of metrics on Σ that varies smoothly with $(s,t) \in D$. We will call any metric of this type a **metric of normal type**.

For all purposes in this paper, we can replace SU(2) by a general compact, connected, and simply connected Lie group G. Now a G-connection on $D \times \Sigma$ is a 1-form $\Xi \in \Omega^1(D \times \Sigma, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g}$. We will write $\mathcal{A}(X)$ for the space of smooth connections over a manifold X, then $\mathcal{A}_{\text{flat}}(X)$ denotes the space of smooth flat connections, and $\mathcal{G}(X)$ is the space of smooth gauge transformations on X (i.e. maps to G). The Sobolev spaces of connections and gauge transformations are denoted by

$$\begin{aligned} \mathcal{A}^{k,p}(X) &= W^{k,p}(X, \mathrm{T}^*X \otimes \mathfrak{g}), \\ \mathcal{G}^{k,p}(X) &= W^{k,p}(X, \mathrm{G}). \end{aligned}$$

We will be dealing with anti-self-dual instantons on $D \times \Sigma$ that satisfy a Lagrangian boundary condition as follows. Let $p > 2$ and fix a handle body Y with boundary $\partial Y = \Sigma$, then the following Lagrangian submanifold is introduced in [W2, Lemma 4.6],

$$\mathcal{L}_Y \;:=\; \mathrm{cl}_{L^p} \left\{ A \in \mathcal{A}_{\text{flat}}(\Sigma) \;\middle|\; \exists \tilde{A} \in \mathcal{A}_{\text{flat}}(Y) : \tilde{A}|_\Sigma = A \right\} \;\subset\; \mathcal{A}^{0,p}(\Sigma).$$

We consider the following boundary value problem for connections $\Xi \in \mathcal{A}(D \times \Sigma)$

$$\left\{ \begin{array}{l} F_\Xi + *F_\Xi = 0, \\ \Xi|_{(s,0)\times\Sigma} \in \mathcal{L}_Y \quad \forall s \in [-r_0, r_0]. \end{array} \right. \tag{2}$$

The compactness result [W3, Thm B] for this boundary value problem can be phrased as follows for the local model. Here $\mathrm{int}(D) = \mathrm{int}(B_{r_0}(0)) \cap \mathbb{H}^2$ denotes the interior in the topology of $\mathbb{H}^2$.

Theorem 1.1 (Compactness) [W3]
Let $p > 2$ and let g^ν be a $\mathcal{C}^\infty$-convergent sequence of metrics of normal type on $D \times \Sigma$. Suppose that $\Xi^\nu \in \mathcal{A}(D \times \Sigma)$ is a sequence of solutions of (2) with respect to the metrics g^ν such that $\|F_{\Xi^\nu}\|_{L^p(D\times\Sigma)}$ is uniformly bounded.

Then there exists a subsequence (again denoted by Ξ^ν) and a sequence of gauge transformations $u^\nu \in \mathcal{G}(D \times \Sigma)$ such that $u^{\nu\,*}\Xi^\nu$ converges uniformly with all derivatives on every compact subset of $\mathrm{int}(D) \times \Sigma$.

Next, we state the energy quantization result that will be proven in section 2.

Theorem 1.2 (Energy quantization)
Let $r_0 > 0$ and let $\mathfrak{m}$ be a $\mathcal{C}^\infty$-compact set of metrics of normal type on $D \times \Sigma$. Then there exists a constant $\varepsilon_0 > 0$ such that the following holds.

Let $\Xi^\nu \in \mathcal{A}(D \times \Sigma)$ be a sequence of solutions of (2) with respect to metrics $g^\nu \in \mathfrak{m}$. Suppose that for all $\delta > 0$

$$\sup_\nu \left\| F_{\Xi^\nu} \right\|_{L^\infty(D_\delta(0)\times\Sigma)} = \infty .$$

Then after taking a subsequence there exist $(s^\nu, t^\nu) \to 0$ and $\varepsilon^\nu \to 0$ such that

$$\int_{D_{\varepsilon^\nu}(s^\nu,t^\nu)\times\Sigma} \left| F_{\Xi^\nu} \right|^2 > \varepsilon_0 .$$

Remark 1.3

(i) By theorem 1.1 the assumptions in theorem 1.2 imply that for a subsequence and with any $p > 2$ one has for all $\delta > 0$

$$\sup_\nu \left\| F_{\Xi^\nu} \right\|_{L^p(D_\delta(0)\times\Sigma)} = \infty .$$

(ii) With the stronger assumption in (i) it suffices to consider a $\mathcal{C}^3$-compact set of metrics in the theorem, as will be seen in the proof. By following through the proof of theorem 1.1, in particular [W3, Thm 2.6], one can moreover check that the set of metrics in theorem 1.2 only needs to be $\mathcal{C}^5$-compact.

To see (i) note that otherwise one would find a sequence Ξ^ν of solutions with respect to a $\mathcal{C}^\infty$-convergent sequence of metrics g^ν and constants C, $\delta > 0$ such that $\|F_{\Xi^\nu}\|_{L^p(D_{2\delta}\times\Sigma)} \leq C$ but $\|F_{\Xi^\nu}\|_{L^\infty(D_\delta\times\Sigma)} \to \infty$. Due to the L^p-bounded curvature one would then find a subsequence and gauges in which the connections converge uniformly on $D_\delta \times \Sigma$. Since the norm of the curvature is gauge invariant, this contradicts the above divergence. In fact, we will need to make the stronger assumption in (i) for some $2 < p < 3$ in order to deduce the energy quantization directly. (This is why we had to establish theorem 1.1 in [W3] in the technically more difficult case $2 < p \leq 4$.)

With this stronger assumption the structure of the proof of theorem 1.2 will be similar to an argument in the interior case, where it is possible to obtain the energy quantization result independently of the removal of singularities and of any geometric knowledge about energies of instantons on S^4. This argument just uses a wellknown mean value inequality for the Laplace operator and will also be explained in section 2. In our case we will need a mean value inequality up to the boundary at which we cannot simply reflect the function. Instead, we will use a mean value inequality for functions with a control on the Laplacian and on the normal derivative at the boundary, which we introduce in [W4]. The following result from [W4] should give an idea of this type of a priori estimate – in the actual proof, we will need a slightly different, more special version.

Lemma 1.4 *For every $n \geq 2$ there exists a constant C such that for all $A, B \geq 0$ there exists $\mu(A,B) > 0$ with the following significance.*

Let $D_r(y) \subset \mathbb{H}^n$ be the Euclidean n-ball in the half space of radius $r > 0$ and center $y \in \mathbb{H}^n$. Suppose that $e \in \mathcal{C}^2(D_r(y), [0,\infty))$ satisfies

$$\begin{cases} \Delta e & \leq B e^{\frac{n+2}{n}}, \\ \frac{\partial}{\partial \nu}\big|_{\partial \mathbb{H}^n} e & \leq A e^{\frac{n+1}{n}}, \end{cases} \qquad \text{and} \qquad \int_{D_r(y)} e < \mu(A,B).$$

Then

$$e(y) \leq C r^{-n} \int_{D_r(y)} e.$$

With the energy quantization established, every sequence of solutions of (1) with bounded energy converges smoothly on the complement of finitely many interior points and boundary slices (modulo gauge and taking a subsequence). Now the remaining key analytic point for the definition of the Floer homology groups is to show that the limit object – after gauge – gives rise to a new solution, that will have less energy. At the interior points, this is Uhlenbeck's removable singularity theorem [U1, Thm 4.1]. For the boundary slices, this requires the following removal of codimension-2-singularities that will be proven in section 5. Here again $D \subset \mathbb{H}^2$ denotes the standard closed half ball with center 0 and some fixed radius $r_0 > 0$, and we introduce the punctured half balls

$$D_r^* := D_r(0) \setminus \{0\}, \qquad D^* := D_{r_0}^* = D \setminus \{0\}.$$

Theorem 1.5 (Removal of singularities for boundary slices) *Let $\Xi \in \mathcal{A}(D^* \times \Sigma)$ be a smooth connection with finite energy $\int_{D^* \times \Sigma} |F_\Xi|^2 < \infty$ and suppose that it satisfies*

$$\begin{cases} *F_\Xi + F_\Xi = 0, \\ \Xi|_{(s,0)\times\Sigma} \in \mathcal{L}_Y \quad \forall s \in [-r_0, 0) \cup (0, r_0]. \end{cases}$$

Then there exists a gauge transformation $u \in \mathcal{G}(D^* \times \Sigma)$ such that $u^*\Xi$ extends to a smooth connection and solution of (2) on $D \times \Sigma$.

Both the energy quantization and the removal of singularities rely on the specific form of the Lagrangian boundary condition: Connections in $\mathcal{L}_Y \subset \mathcal{A}^{0,p}(\Sigma)$ are extended from $\partial Y = \Sigma$ to flat connections on Y with the L^2-norm on Σ controlling the L^3-norm on Y. The corresponding linear and nonlinear extension results are given in the following lemma and are proven in section 3.

Lemma 1.6 *There exists a constant C_Y such that the following holds.*

(i) For every smooth path $A : (-\varepsilon, \varepsilon) \to \mathcal{L}_Y \cap \mathcal{A}(\Sigma)$ there exists another path $\tilde{A} : (-\varepsilon, \varepsilon) \to \mathcal{A}_{\mathrm{flat}}(Y)$ with $\partial_s \tilde{A}(0)|_{\partial Y} = \partial_s A(0)$ such that

$$\|\partial_s \tilde{A}(0)\|_{L^3(Y)} \leq C_Y \|\partial_s A(0)\|_{L^2(\Sigma)}.$$

(ii) For all $A_0, A_1 \in \mathcal{L}_Y \cap \mathcal{A}(\Sigma)$ there exist $\tilde{A}_0, \tilde{A}_1 \in \mathcal{A}_{\mathrm{flat}}(Y)$ with $A_i = \tilde{A}_i|_{\partial Y}$ such that

$$\|\tilde{A}_0 - \tilde{A}_1\|_{L^3(Y)} \leq C_Y \|A_0 - A_1\|_{L^2(\Sigma)}. \tag{3}$$

Remark 1.7 *The constant C_Y in lemma 1.6 can be chosen uniform for a $\mathcal{C}^0$-neighbourhood of metrics on Y and the induced metrics on $\Sigma = \partial Y$.*

This can be seen by using a fixed metric for the construction of the extensions. The $L^2(\Sigma)$- and $L^3(Y)$-norms for different metrics are then equivalent with a small factor for $\mathcal{C}^0$-close metrics.

The nonlinear extension in (ii) allows to define a local Chern-Simons functional for short arcs from $\mathcal{L}_Y$ to $\mathcal{L}_Y$: We consider smooth paths $A : [0,\pi] \to \mathcal{A}(\Sigma)$ with endpoints $A(0), A(\pi) \in \mathcal{L}_Y$. For such paths lemma 1.6 (ii) provides extensions $\tilde{A}(0), \tilde{A}(\pi) \in \mathcal{A}_{\text{flat}}(Y)$ of $A(0), A(\pi)$ that satisfy (3). We pick any such extensions to define

$$\begin{aligned} \mathcal{CS}(A) := &-\tfrac{1}{2}\int_0^\pi \int_\Sigma \langle\, A \wedge \partial_\phi A \,\rangle \,\mathrm{d}\phi \\ &+ \tfrac{1}{12}\int_Y \langle\, \tilde{A}(0) \wedge [\tilde{A}(0) \wedge \tilde{A}(0)] \,\rangle - \langle\, \tilde{A}(\pi) \wedge [\tilde{A}(\pi) \wedge \tilde{A}(\pi)] \,\rangle. \end{aligned} \tag{4}$$

Here the notations $[\cdot\cdot]$ and $\langle\,\cdot\cdot\,\rangle$ indicate that the values of the differential forms are paired via the Lie bracket and an equivariant inner product on $\mathfrak{g}$ respectively. This is the actual Chern-Simons functional on $\bar{Y} \cup_{\{0\}\times\Sigma} [0,\pi]\times\Sigma \cup_{\{\pi\}\times\Sigma} Y$ of the connection given by $\tilde{A}(0)$, A, and $\tilde{A}(\pi)$ on the different parts. (Here $\bar{Y}$ denotes Y with the reversed orientation.) The extensions $\tilde{A}(0)$ and $\tilde{A}(\pi)$ could both vary by gauge transformations that are trivial on $\partial Y = \Sigma$. So the connection on the above closed manifold might also vary by a gauge transformation (that is trivial on the middle part). The Chern-Simons functional however does not vary under gauge transformations that are homotopic to $\mathbb{1}$, and it only changes by multiples of $4\pi^2$ for others.[2] In fact, if we restrict to short paths, then we will see in section 4 that our local Chern-Simons functional is welldefined and satisfies an isoperimetric inequality.

Lemma 1.8 (Isoperimetric inequality)
There exists $\varepsilon > 0$ such that for all smooth paths $A : [0,\pi] \to \mathcal{A}(\Sigma)$ with $A(0), A(\pi) \in \mathcal{L}_Y$ and $\int_0^\pi \|\partial_\phi A\|_{L^2(\Sigma)} \leq \varepsilon$ the local Chern-Simons functional (4) is welldefined and satisfies

$$|\mathcal{CS}(A)| \;\leq\; \left(\int_0^\pi \big\|\partial_\phi A\big\|_{L^2(\Sigma)} \,\mathrm{d}\phi \right)^2 .$$

The significance of the local Chern-Simons functional for theorem 1.5 is in the fact that the energy of the connection can be expressed by this functional. The isoperimetric inequality will then provide a control on the rate of decay of the energy on small neighbourhoods of the singularity. This can be combined with mean value inequalities as in lemma 1.4 to obtain estimates on the connection (in a specific gauge) near the singularity. Finally, we will be able to remove the singularity using a cutoff construction and the compactness result, theorem 1.1.

[2] This constant is correct for $\mathrm{G} = \mathrm{SU}(2)$ with $\langle\, \xi, \eta \,\rangle = \mathrm{tr}(\xi^*\eta)$. For a general Lie group we can achieve the same constant by scaling the inner product appropriately.

Note that in our approach all bubbling at the boundary is treated globally, even if it could be described as an instanton on S^4 bubbling off at the boundary. In fact, the energy quantization result also holds for interior slices (i.e. $\{s\} \times \{t\} \times \Sigma \subset \mathbb{R} \times \operatorname{int} M$ in a tubular neighbourhood $\mathbb{R} \times [0, \varepsilon) \times \Sigma$ of $\mathbb{R} \times \partial M$). This description of the bubbling phenomena would then require a removable singularity result for anti-self-dual instantons with a singularity of codimension 2. An obviously necessary condition for this result is that the limit holonomy around the singularity vanishes almost everywhere. It was shown by Sibner-Sibner [Si, Thm 5.2] and Rade [R, Thm 2.1] that this condition is in fact sufficient. Moreover, the fact that interior bubbling only occurs at isolated points shows that the holonomy condition is satisfied at interior slices. This is of little use in our context, so we stick to a pointwise description of interior bubbling.

However, our techniques for the removal of slice singularities at the boundary also give rise to an alternative approach to the Sibner-Sibner result for interior slices. In fact, this approach might lead to a general normal form in terms of the limit holonomy for finite energy anti-self-dual instantons with a singularity of codimension 2. (This question was raised by Kronheimer and Mrowka in [KM].) However, in this paper, we only consider a special case in which we obtain a largely simplified proof of the removal of singularities. This proof is given in section 5. In order to state the result we denote by B the standard closed ball with center 0 and some fixed radius $r_0 > 0$, and we introduce the punctured ball B^*,

$$B := B_{r_0}(0) \subset \mathbb{R}^2, \qquad\qquad B^* := B \setminus \{0\}.$$

Introducing polar coordinates (r, ϕ) on B^* one can write any connection on $B^* \times \Sigma$ in the form $\Xi = R\mathrm{d}r + \Phi\mathrm{d}\phi + A$, where A is a family of 1-forms on Σ. The holonomy condition in [Si] is equivalent to the existence of a gauge in which

$$\int_0^{2\pi} \left\|\Phi(r,\phi)\right\|^2_{L^2(\Sigma)} \mathrm{d}\phi \underset{r\to 0}{\longrightarrow} 0.$$

We will make the stronger assumption that in fact there is a gauge in a neighbourhood of the singular slice in which $\Phi \equiv 0$.

Remark 1.9 (Removal of singularities for interior slices) [Si, R] *Let $\Xi \in \mathcal{A}(B^* \times \Sigma)$ be a smooth anti-self-dual connection with finite energy $\int_{B^*\times\Sigma} |F_\Xi|^2 < \infty$ and suppose that Ξ is gauge equivalent to a connection on $B^* \times \Sigma$ with $\Phi \equiv 0$. Then there exists a gauge transformation $u \in \mathcal{G}(B^* \times \Sigma)$ such that $u^*\Xi$ extends to a smooth anti-self-dual connection on $B \times \Sigma$.*

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2 Energy quantization

The energy quantization result for anti-self-dual instantons at interior points could be phrased as follows (in the special case of a Euclidean metric).

Theorem 2.1 *There exists a constant $\varepsilon_0 > 0$ such that the following holds.*

Let $B := B_{r_0}(0) \subset \mathbb{R}^4$ be the Euclidean 4-ball of radius $r_0 > 0$ and let $\Xi^\nu \in \mathcal{A}(B)$ be a sequence of anti-self-dual connections. Suppose that

$$\sup_\nu \left\| F_{\Xi^\nu} \right\|_{L^\infty(B_\delta(0))} \;=\; \infty \qquad\qquad \forall \delta > 0.$$

Then after taking a subsequence there exist $B \ni x^\nu \to 0$ and $\varepsilon^\nu \to 0$ such that for all $\nu \in \mathbb{N}$

$$\int_{B_{\varepsilon^\nu}(x^\nu)} \left| F_{\Xi^\nu} \right|^2 \;>\; \varepsilon_0 .$$

This is of course a wellknown result in gauge theory. Here we give a purely analytic proof that does not use the removable singularity result. This exhibits a general method for establishing energy quantization whenever one has a (non-linear) bound on the Laplacian of the energy density, and this implies a mean value inequality on balls of small energy. In our case, this mean value inequality will be provided by the following wellknown result (see e.g. [W4]).

Proposition 2.2 *For every $n \in \mathbb{N}$ there exist constants C, $\mu > 0$, and $\delta > 0$ such that the following holds.*

Let $\mathbb{R}^n$ be equipped with a metric g such that $\|g - \mathbb{1}\|_{W^{1,\infty}} \leq \delta$. Let $B_r(0) \subset \mathbb{R}^n$ be the geodesic ball of radius $0 < r \leq 1$. Suppose that $e \in \mathcal{C}^2(B_r(0), [0, \infty))$ satisfies for some $A, B \geq 0$

$$\Delta e \leq Ae + Be^{\frac{n+2}{n}} \qquad \text{and} \qquad \int_{B_r(0)} e < \mu B^{-\frac{n}{2}} .$$

Then

$$e(0) \leq C\big(A^{\frac{n}{2}} + r^{-n}\big) \int_{B_r(0)} e .$$

Proof of theorem 2.1: By assumption one can find a subsequence and points $B \ni x^\nu \to 0$ such that $R^\nu := |F_{\Xi^\nu}(x^\nu)|^{\frac{1}{2}} \to \infty$. We pick a sequence $\varepsilon^\nu \to 0$ such that still $\varepsilon^\nu R^\nu \to \infty$. Now consider the energy density functions $e^\nu = |F_{\Xi^\nu}|^2 : B \to [0, \infty)$. One can check (see (5) below) that $\Delta e^\nu \leq 8(e^\nu)^{\frac{3}{2}}$. Let $\mu > 0$ be the constant from the mean value inequality proposition 2.2, then the theorem holds with $\varepsilon_0 = \frac{\mu}{64}$. Indeed, for all sufficiently large $\nu \in \mathbb{N}$ (such that $B_{\varepsilon^\nu}(x^\nu) \subset B$) we either have $\int_{B_{\varepsilon^\nu}(x^\nu)} e^\nu > \varepsilon_0$, or by means of proposition 2.2

$$(R^\nu)^4 \;=\; e^\nu(x^\nu) \;\leq\; C(\varepsilon^\nu)^{-4} \int_{B_{\varepsilon^\nu}(x^\nu)} e^\nu$$

and thus $(\varepsilon^\nu R^\nu)^4 \leq C\varepsilon_0$. Since $\varepsilon^\nu R^\nu \to \infty$ the latter can only be true for finitely many $\nu \in \mathbb{N}$. $\square$

The proof of theorem 1.2 will run along similar lines. Here the mean value inequality (with a boundary condition) will be applied to the functions $\|F_{\Xi^\nu}\|^2_{L^2(\Sigma)}$ that are defined on $D = D_{r_0}(0) \subset \mathbb{H}^2$. So firstly, we need to show that the assumption in theorem 1.2, i.e. no local uniform bound for the curvature near the slice $\{0\} \times \Sigma$, actually implies a blowup of the above function (the slicewise L^2-norm of the curvature) at $0 \in \mathbb{H}^2$. Here remark 1.3 (i) is crucial: It asserts that in fact there is no local L^p-bound for the curvature near $\{0\} \times \Sigma$ for any $p > 2$. From this stronger assumption (we need $p < 3$), lemma 2.4 below will then imply the blowup of $\|F_{\Xi^\nu}\|^2_{L^2(\Sigma)}$.

The underlying analytic facts of this lemma and the whole proof of theorem 1.2 will be mean value inequalities for both $\|F_{\Xi^\nu}\|^2_{L^2(\Sigma)}$ (on a 2-dimensional domain with boundary) and $|F_{\Xi^\nu}|^2$ (on a 4-dimensional domain). So we shall first calculate the Laplacians and normal derivatives of these functions. For that purpose we write the connection in the splitting

$$\Xi = A + \Phi \mathrm{d}s + \Psi \mathrm{d}t,$$

where $A : D \to \Omega^1(\Sigma, \mathfrak{g})$ and $\Phi, \Psi : D \to \Omega^0(\Sigma, \mathfrak{g})$.[3] By d_A and d_A^* we then denote the families (parametrized by $(s,t) \in D$) of operators on Σ corresponding to $A(s,t)$. Moreover, we introduce the covariant derivatives

$$\nabla_s := \partial_s + [\Phi, \cdot], \qquad \nabla_t := \partial_t + [\Psi, \cdot].$$

Now the components of the curvature are F_A and

$$\begin{aligned} B_s := \; \partial_s A - \mathrm{d}_A \Phi \;&=\; [\nabla_s, \mathrm{d}_A], \\ B_t := \; \partial_t A - \mathrm{d}_A \Psi \;&=\; [\nabla_t, \mathrm{d}_A], \\ \partial_t \Phi - \partial_s \Psi + [\Psi, \Phi] \;&=\; [\nabla_t, \nabla_s]. \end{aligned}$$

The Bianchi identity $\mathrm{d}_\Xi F_\Xi = 0$ becomes in this splitting

$$\nabla_s F_A = \mathrm{d}_A B_s, \qquad \nabla_t F_A = \mathrm{d}_A B_t, \qquad \nabla_s B_t - \nabla_t B_s = \mathrm{d}_A [\nabla_t, \nabla_s],$$

and the anti-self-duality equation is

$$*B_s = B_t, \qquad *F_A = [\nabla_t, \nabla_s].$$

Lemma 2.3 *There is a constant C (varying continuously with the metric of normal type in the $\mathcal{C}^2$-topology) such that for all solutions $\Xi \in \mathcal{A}(D \times \Sigma)$ of (2)*

$$\begin{aligned} \Delta \big|F_\Xi\big|^2 \;&\leq\; C \big|F_\Xi\big|^2 + 8 \big|F_\Xi\big|^3, \\ \Delta \big\|F_\Xi\big\|^2_{L^2(\Sigma)} \;&\leq\; C \big\|F_\Xi\big\|^2_{L^2(\Sigma)} - 20 \langle\, F_A \,,\, [B_s \wedge B_s] \,\rangle_{L^2(\Sigma)} \\ &\leq\; C \big(1 + \big\|F_A\big\|_{L^\infty(\Sigma)}\big) \big\|F_\Xi\big\|^2_{L^2(\Sigma)}, \\ -\tfrac{\partial}{\partial t}\big|_{t=0} \big\|F_\Xi\big\|^2_{L^2(\Sigma)} \;&\leq\; C \big\|B_s\big\|^2_{L^2(\Sigma)} - 4 \int_\Sigma \langle\, \nabla_s B_s \wedge B_s \,\rangle \\ &\leq\; C \big(\big\|B_s\big\|^2_{L^2(\Sigma)} + \big\|B_s\big\|^3_{L^2(\Sigma)} \big). \end{aligned}$$

[3] Note that this notation differs from [W3], where we wrote $A = B + \Phi \mathrm{d}s + \Psi \mathrm{d}t$.

Proof: The anti-self-duality equation together with the Bianchi identity gives

$$\begin{aligned}\nabla_s B_s + \nabla_t B_t &= *\big(-\nabla_s B_t + \nabla_t B_s\big) - (\partial_s *)B_t + (\partial_t *)B_s \\ &= - * \mathrm{d}_A * F_A - (\partial_s *)B_t + (\partial_t *)B_s.\end{aligned}$$

Using this identity we obtain

$$\begin{aligned}&\big(\nabla_s^2 + \nabla_t^2\big)B_s \\ &= \nabla_s\big(-\nabla_t B_t - *\mathrm{d}_A * F_A - (\partial_s *)B_t + (\partial_t *)B_s\big) + \nabla_t\big(\nabla_s B_t - \mathrm{d}_A * F_A\big) \\ &= [*F_A, B_t] - *\mathrm{d}_A * \nabla_s F_A - *[B_s, *F_A] - \mathrm{d}_A * \nabla_t F_A - [B_t, *F_A] \\ &\quad - (\partial_s *)\big(\mathrm{d}_A * F_A + \nabla_s B_t\big) + (\partial_t *)\nabla_s B_s - *\mathrm{d}_A(\partial_s *)F_A - \mathrm{d}_A(\partial_t *)F_A \\ &\quad - (\partial_s^2 *)B_t + (\partial_s\partial_t *)B_s \\ &= \mathrm{d}_A^*\mathrm{d}_A B_s + \mathrm{d}_A\mathrm{d}_A^* B_s - 3 * [B_s, *F_A] - (\partial_s^2 *)B_t + (\partial_s\partial_t *)B_s \\ &\quad - (\partial_s *)\nabla_t B_s + (\partial_t *)\nabla_s B_s - *\mathrm{d}_A(\partial_s *)F_A - \mathrm{d}_A(\partial_t *)F_A,\end{aligned}$$

$$\begin{aligned}\big(\nabla_s^2 + \nabla_t^2\big)F_A &= \nabla_s \mathrm{d}_A B_s + \nabla_t \mathrm{d}_A B_t \\ &= \mathrm{d}_A\big(\nabla_s B_s + \nabla_t B_t\big) + [B_s \wedge B_s] + [B_t \wedge B_t] \\ &= \mathrm{d}_A\mathrm{d}_A^* F_A + 2[B_s \wedge B_s].\end{aligned}$$

Continuing these calculations leads to the Bochner-Weitzenböck formula (c.f. [BL, Thm 3.10]) for anti-self-dual connections

$$0 \;=\; \big(\mathrm{d}_\Xi\mathrm{d}_\Xi^* + \mathrm{d}_\Xi^*\mathrm{d}_\Xi\big)F_\Xi \;=\; \nabla_\Xi^*\nabla_\Xi F_\Xi + F_\Xi \circ (\mathrm{Ric} \wedge g + 2R) + \mathcal{R}^\Xi(F_\Xi).$$

The quadratic term $\mathcal{R}^\Xi(F_\Xi) \in \Omega^2(D \times \Sigma, \mathfrak{g})$ can be expressed with the help of a local orthonormal frame $(e_1, \ldots, e_4)$ of $\mathrm{T}(D \times \Sigma)$ as

$$\mathcal{R}^\Xi(F_\Xi)(X,Y) = \sum_{j=1}^{4}\big\{[F_\Xi(e_j,X), F_\Xi(e_j,Y)] - [F_\Xi(e_j,Y), F_\Xi(e_j,X)]\big\}.$$

This gives the first estimate

$$\begin{aligned}\Delta\big|F_\Xi\big|^2 &= -2\big|\nabla_\Xi F_\Xi\big|^2 + 2\langle\, F_\Xi\,,\, \nabla_\Xi^*\nabla_\Xi F_\Xi\,\rangle \\ &\le -2\langle\, F_\Xi\,,\, F_\Xi \circ (\mathrm{Ric}\wedge g + 2R)\,\rangle - 2\langle\, F_\Xi\,,\, \mathcal{R}^\Xi(F_\Xi)\,\rangle \qquad (5) \\ &\le C\big|F_\Xi\big|^2 + 8\big|F_\Xi\big|^3.\end{aligned}$$

Here the constant C depends on the Ricci transform Ric and the scalar curvature R of the metric g. It can thus be chosen uniform for a $\mathcal{C}^2$-neighbourhood of the fixed metric.

The purpose of the calculations in the beginning is the following identity:

$$\begin{aligned}-\tfrac{1}{4}\Delta\big\|F_\Xi\big\|_{L^2(\Sigma)}^2 &= \big\|\nabla_s F_A\big\|_{L^2}^2 + \big\|\nabla_t F_A\big\|_{L^2}^2 + \big\|\nabla_s B_s\big\|_{L^2}^2 + \big\|\nabla_t B_s\big\|_{L^2}^2 \\ &\quad + \langle\, F_A\,,\, \big(\nabla_s^2 + \nabla_t^2\big)F_A\,\rangle_{L^2(\Sigma)} + \langle\, B_s\,,\, \big(\nabla_s^2 + \nabla_t^2\big)B_s\,\rangle_{L^2(\Sigma)} \\ &\quad + \langle\, *F_A\,,\, (\partial_s^2 *)F_A\,\rangle_{L^2(\Sigma)} + \langle\, *B_s\,,\, (\partial_s^2 *)B_s\,\rangle_{L^2(\Sigma)} \\ &\quad + \langle\, (\partial_s *)F_A\,,\, *\nabla_s F_A\,\rangle_{L^2(\Sigma)} + \langle\, (\partial_s *)B_s\,,\, *\nabla_s B_s\,\rangle_{L^2(\Sigma)}\end{aligned}$$

$$
\begin{aligned}
&= \big\|\nabla_s B_s\big\|^2_{L^2(\Sigma)} + \big\|\nabla_t B_s\big\|^2_{L^2(\Sigma)} + \big\|\mathrm{d}_A B_s\big\|^2_{L^2(\Sigma)} + \big\|\mathrm{d}_A^* B_s\big\|^2_{L^2(\Sigma)} \\
&\quad + \big\|\nabla_s F_A\big\|^2_{L^2(\Sigma)} + \big\|\nabla_t F_A\big\|^2_{L^2(\Sigma)} + \big\|\mathrm{d}_A^* F_A\big\|^2_{L^2(\Sigma)} + 5\langle\, F_A\,,\, [B_s\wedge B_s]\,\rangle_{L^2(\Sigma)} \\
&\quad - \langle\, B_s\,,\, *(\partial_s^2 *)B_s + (\partial_s^2 *) * B_s - (\partial_s\partial_t *)B_s\,\rangle_{L^2(\Sigma)} + \langle\, *F_A\,,\, (\partial_s^2 *)F_A\,\rangle_{L^2(\Sigma)} \\
&\quad + \langle\, (\partial_s *)B_s\,,\, \nabla_t B_s + *\nabla_s B_s\,\rangle_{L^2(\Sigma)} - \langle\, (\partial_t *)B_s\,,\, \nabla_s B_s\,\rangle_{L^2(\Sigma)} \\
&\quad + 2\langle\, \mathrm{d}_A B_s\,,\, *(\partial_s *)F_A\,\rangle_{L^2(\Sigma)} - \langle\, \mathrm{d}_A^* B_s\,,\, (\partial_t *)F_A\,\rangle_{L^2(\Sigma)}.
\end{aligned}
$$

This yields the second inequality

$$
\Delta\big\|F_\Xi\big\|^2_{L^2(\Sigma)} \le C\big(\big\|B_s\big\|^2_{L^2(\Sigma)} + \big\|F_A\big\|^2_{L^2(\Sigma)}\big) - 20\langle\, F_A\,,\, [B_s\wedge B_s]\,\rangle_{L^2(\Sigma)}.
$$

Here the constant C depends on the second derivatives of $g_{s,t}$ and its inverse. Using the Bianchi identity, the anti-self-duality equation, and in addition the boundary condition $F_A\big|_{t=0} = 0$ we obtain for the normal derivative as claimed

$$
\begin{aligned}
-\tfrac14 \tfrac{\partial}{\partial t}\big|_{t=0}\big\|F_\Xi\big\|^2_{L^2(\Sigma)} &= -\big(\langle\, F_A\,,\, \nabla_t F_A\,\rangle_{L^2(\Sigma)} + \langle\, B_s\,,\, \nabla_t B_s\,\rangle_{L^2(\Sigma)}\big)\big|_{t=0} \\
&= \langle\, B_s\,,\, -\nabla_s B_t + \mathrm{d}_A * F_A\,\rangle_{L^2(\Sigma)}\big|_{t=0} \\
&= \int_\Sigma \langle\, B_s\wedge\big(\nabla_s B_s - *(\partial_s *)B_s\big)\,\rangle\big|_{t=0} \\
&\le \Big(C\big\|B_s\big\|^2_{L^2(\Sigma)} - \int_\Sigma \langle\, \nabla_s B_s\wedge B_s\,\rangle\Big)\Big|_{t=0}.
\end{aligned}
$$

The second estimate for the normal derivative can be checked in any gauge at a fixed $(s_0,0)\in D\cap\partial\mathbb{H}^2$. We choose a gauge with $\Phi\equiv 0$ and hence $B_s = \partial_s A$. Then for the path $\Xi|_{(\cdot,0)\times\Sigma} = A(\cdot,0)$ in $\mathcal{L}_Y$ lemma 1.6 (i) provides a path of extensions $\tilde A : (s_0-\varepsilon, s_0+\varepsilon)\to\mathcal{A}_{\rm flat}(Y)$ such that $\partial_s\tilde A(s_0)|_\Sigma = \partial_s A(s_0,0)$ and $\|\partial_s\tilde A(s_0)\|_{L^3(Y)} \le C\|\partial_s A(s_0,0)\|_{L^2(\Sigma)}$. Here we fix a smooth path of metrics on Y that extend the metrics $g_{s,0}$ on Σ for $s\in[-r_0,r_0]$. The constant C can then be chosen uniform for all $(s_0,0)\in D\cap\partial\mathbb{H}^2$. So we calculate at $s=s_0$

$$
\begin{aligned}
-\int_\Sigma \langle\, \nabla_s B_s\wedge B_s\,\rangle &= \int_{\partial Y} \langle\, \partial_s\tilde A\wedge\partial_s\partial_s\tilde A\,\rangle \\
&= \int_Y \langle\, \mathrm{d}_{\tilde A}\partial_s\tilde A\wedge\partial_s^2\tilde A\,\rangle - \int_Y \langle\, \partial_s\tilde A\wedge\mathrm{d}_{\tilde A}\partial_s^2\tilde A\,\rangle \\
&= \int_Y \langle\, \partial_s\tilde A\wedge[\partial_s\tilde A\wedge\partial_s\tilde A]\,\rangle \\
&\le \|\partial_s\tilde A\|^3_{L^3(Y)} \;\le\; C^3\|\partial_s A\|^3_{L^2(\Sigma)} \;=\; C^3\|B_s\|^3_{L^2(\Sigma)}.
\end{aligned}
$$

Here we have used the fact that $F_{\tilde A}\equiv 0$, so $\mathrm{d}_{\tilde A}\partial_s\tilde A = \partial_s F_{\tilde A} = 0$ and

$$
0 \;=\; \partial_s^2 F_{\tilde A} \;=\; \mathrm{d}_{\tilde A}\partial_s^2\tilde A + [\partial_s\tilde A\wedge\partial_s\tilde A]. \qquad \square
$$

The significance of the following lemma is that a uniform bound on the slicewise L^2-norm of the curvature of an anti-self-dual connection implies an

L^p-bound on the curvature for any $p < 3$. The specific value of the latter bound is not relevant here. We only give it for comparison with a similar calculation in the proof of proposition 2.7.

Lemma 2.4 *Fix $r_0 > 0$, let $2 < p < 3$, and let $\mathfrak{m}$ be a $\mathcal{C}^1$-compact set of metrics of normal type on $D \times \Sigma$. Then there exists a constant C_p such that the following holds for all $0 < \delta \le \frac{1}{2} r_0$.*

Let $\Xi \in \mathcal{A}(D_{2\delta}(0) \times \Sigma)$ be anti-self-dual with respect to a metric in $\mathfrak{m}$ and suppose that for some constant c

$$\big\|F_\Xi(s,t)\big\|_{L^2(\Sigma)} \le c \qquad \forall (s,t) \in D_{2\delta}(0).$$

Then

$$\big\|F_\Xi\big\|_{L^p(D_\delta(0)\times\Sigma)} \;\le\; C_p\big(\delta^{\frac{4}{p}-1}\, c + \delta^{\frac{2}{p}}\, c^{p-\frac{2}{p}}\big).$$

Proof: Fix a metric of normal type on $D \times \Sigma$. It suffices to prove the estimate with a uniform constant for all metrics of normal type in a $\mathcal{C}^1$-neighbourhood of the fixed metric. We choose this neighbourhood such that we have a uniform constant C_1 in the estimate from lemma 2.3,

$$\Delta\big|F_\Xi\big|^2 \le C_1\big|F_\Xi\big|^2 + 8\big|F_\Xi\big|^3.$$

Next, the normal coordinates at any $(s,t,z) \in D_{\frac{1}{2}r_0}(0) \times \Sigma$ give a coordinate chart on $B_R(s,t,0,0) \cap \mathbb{H}^4$ with $R > 0$ in which the fixed metric (and hence all metrics in a sufficiently small neighbourhood) is $\mathcal{C}^1$-close to the Euclidean metric. This $R > 0$ can be chosen uniform for all $(s,t,z) \in D_{\frac{1}{2}r_0}(0) \times \Sigma$ such that the metrics in the coordinates meet the assumption of proposition 2.2. Now let $\bar{\mu} := \frac{\mu}{64}$ where $\mu > 0$ is the constant from the theorem, and assume that $(s,t,z) \in D_\delta(0) \times \Sigma$. One can then apply this mean value inequality to $e = |F_\Xi|^2$ on $B_r(s,t,0,0)$ for $r = \min\big(t, R, c^{-1}\sqrt{\bar{\mu}/\pi}\,\big)$. Since

$$\int_{B_r(s,t,0,0)} \big|F_\Xi\big|^2 \;\le\; \int_{B_r(s,t)} \big\|F_\Xi\big\|_{L^2(\Sigma)}^2 \;\le\; \pi\, r^2 c^2 \;\le\; \bar{\mu}$$

we obtain with a uniform constant C for all $(s,t,z) \in D_\delta(0) \times \Sigma$

$$\big|F_\Xi(s,t,z)\big|^2 \;\le\; C r^{-4} \int_{B_r(s,t,0,0)} \big|F_\Xi\big|^2 \;\le\; C\pi c^2 \min\big(t, R, c^{-1}\sqrt{\bar{\mu}/\pi}\,\big)^{-2}.$$

(Here we have used the fact that $r \le R$, so ${C_1}^2 + r^{-4} \le C r^{-4}$ with a uniform constant depending on R.) This pointwise control of F_Ξ combines with the bound on $\|F_\Xi(s,t)\|_{L^2(\Sigma)}$ to yield for $2 < p < 3$

$$\begin{aligned}
\big\|F_\Xi\big\|^p_{L^p(D_\delta(0)\times\Sigma)} &\le \int_{D_\delta(0)} \big\|F_\Xi\big\|^{p-2}_{L^\infty(\Sigma)} \big\|F_\Xi\big\|^2_{L^2(\Sigma)} \mathrm{d}s\mathrm{d}t \\
&\le \delta c^2 \int_0^\delta (C\pi c^2)^{\frac{p-2}{2}} \big(t^{2-p} + \min\big(R, c^{-1}\sqrt{\bar{\mu}/\pi}\,\big)^{2-p}\big)\, \mathrm{d}t \\
&\le (C\pi)^{\frac{p-2}{2}} \delta c^p \Big(\tfrac{1}{3-p}\delta^{3-p} + \delta R^{2-p} + \delta c^{p-2}\big(\tfrac{\bar{\mu}}{\pi}\big)^{\frac{2-p}{2}}\Big) \\
&\le {C_p}^p \big(\delta^{4-p}\, c^p + \delta^2\, c^{2p-2}\big).
\end{aligned}$$

$\square$

Note that the assumption $p < 3$ is crucial in this estimate. So a pointwise blowup of the curvature is not enough to deduce a blowup of $\|F_\Xi\|_{L^2(\Sigma)}$. As a consequence, it is essential that the compactness result [W3, Thm B] for solutions of (2) with an L^p-bound on the curvature was established for $2 < p \leq 4$ (as well as for the easier case $p > 4$). These results put us in the following position near any slice of the boundary: There either is a local L^p-bound with $2 < p < 3$ for the curvature (and hence a convergent subsequence up to gauge) or a blowup of the functions $\|F_{\Xi^\nu}\|^2_{L^2(\Sigma)} : D \to [0, \infty)$.

If one now tries to mimic the proof of theorem 2.1, one firstly needs the following mean value inequality for the Laplacian with Neumann boundary condition, a proof of which can be found in [W4].

Proposition 2.5 *There exist constants C, $\mu > 0$ such that the following holds.*

Let $D_r(y) \subset \mathbb{H}^2$ be a Euclidean ball of radius $r > 0$ and center $y \in \mathbb{H}^2$ intersected with the half space. Suppose that $e \in \mathcal{C}^2(D_r(y), [0, \infty))$ satisfies for some constants $A, B \geq 0$

$$\left\{ \begin{array}{ll} \Delta e & \leq Be, \\ -\frac{\partial}{\partial t}\big|_{\partial \mathbb{H}^2} e & \leq A\bigl(e + e^{\frac{3}{2}}\bigr), \end{array} \right. \qquad \textit{and} \qquad \int_{D_r(y)} e < \mu A^{-2}.$$

Then

$$e(y) \leq C\bigl(B + A^2 + r^{-2}\bigr) \int_{D_r(y)} e.$$

Another ingredient in our proof of energy quantization is the Hofer trick, [HZ, 6.4 Lemma 5], which we state here for convenience.

Lemma 2.6 (Hofer trick) *Let $f : X \to [0, \infty)$ be continuous on the complete metric space X. Then for every $x_0 \in X$ and $\varepsilon_0 > 0$ there exist $x \in B_{2\varepsilon_0}(x_0) \subset X$ and $0 < \varepsilon \leq \varepsilon_0$ such that $\varepsilon f(x) \geq \varepsilon_0 f(x_0)$ and $f(y) \leq 2f(x)$ for all $y \in B_\varepsilon(x)$.*

The assumptions of proposition 2.5 will be verified by lemma 2.3. Firstly, the estimate for the normal derivative at the boundary, $-\frac{\partial}{\partial t}\big|_{t=0} e^\nu$, results from lemma 1.6 (i), i.e. from a (linear) extension of tangent vectors to $\mathcal{L}_Y$ to 1-forms on Y. Secondly, one should note that the term $\langle F_A \, , \, [B_s \wedge B_s] \rangle_{L^2(\Sigma)}$ in the expression for $\Delta \|F_\Xi\|^2_{L^2(\Sigma)}$ in lemma 2.3 is not yet in a form that can be controlled by any power of $\|F_\Xi\|^2_{L^2(\Sigma)}$ as required above. This is the central analytic problem of the bubbling analysis. It is overcome by the following proposition which shows that $\|F_A\|_{L^\infty(\Sigma)}$ is essentially bounded by $\|F_\Xi\|^2_{L^2(\Sigma)}$.

If this bound was not true, then one would roughly find a pointwise blowup of the F_A-component of the curvature while the energy goes to zero. A local rescaling would then lead to a nonflat limit connection in contradiction to the vanishing of the energy. The nontrivial limit is obtained only when the blowup is mainly in the F_A-component of the curvature. This is since after the local rescaling one has $\mathcal{C}^0$-convergence only for F_A (which satisfies a Dirichlet boundary condition) and not for B_s (for which the global Lagrangian boundary condition is lost). We will first state this result and show how it leads to a proof of theorem 1.2, and then give its actual proof.

Recall that the boundary value problem (2) is the anti-self-duality equation together with a Lagrangian boundary condition in the space of flat connections over Σ. For the proposition below, it would actually suffice to assume only the flat boundary condition $F_\Xi|_{(s,0)\times\Sigma} = 0$ in (2).

Proposition 2.7

(i) Let $\Xi^\nu \in \mathcal{A}(D\times\Sigma)$ be a sequence of solutions of (2) such that for some $\frac{1}{2}r_0 \geq \delta > 0$

$$\sup_\nu \sup_{(s,t)\in D_{2\delta}} \bigl\| F_{\Xi^\nu}(s,t) \bigr\|_{L^2(\Sigma)} < \infty.$$

Then

$$\sup_\nu \sup_{(s,t)\in D_{\delta}} \bigl\| F_{A^\nu}(s,t) \bigr\|_{L^\infty(\Sigma)} < \infty.$$

(ii) For every $\mathcal{C}^3$-compact set $\mathfrak{m}$ of metrics of normal type on $D\times\Sigma$ and every $\Delta > 0$ there exists a constant C such that the following holds:

Let $\Xi^\nu \in \mathcal{A}(D\times\Sigma)$ be a sequence of solutions of (2) with respect to metrics $g^\nu \in \mathfrak{m}$. Suppose that $(s^\nu,t^\nu) \in D_{\frac{1}{2}r_0}$, $\varepsilon^\nu \to 0$, and $R^\nu \to \infty$ such that $\varepsilon^\nu R^\nu \geq \Delta > 0$ for all $\nu\in\mathbb{N}$ and

$$\bigl\| F_{\Xi^\nu}(s,t) \bigr\|_{L^2(\Sigma)} \leq R^\nu \qquad \forall (s,t) \in D_{2\varepsilon^\nu}(s^\nu,t^\nu).$$

Then for all sufficiently large $\nu\in\mathbb{N}$

$$\bigl\| F_{A^\nu}(s,t) \bigr\|_{L^\infty(\Sigma)} \leq C\bigl(R^\nu\bigr)^2 \qquad \forall (s,t) \in D_{\varepsilon^\nu}(s^\nu,t^\nu).$$

Proof of theorem 1.2:
Let $\mathfrak{m}$ be a $\mathcal{C}^3$-compact set of metrics of normal type on $D\times\Sigma$ and consider a sequence $\Xi^\nu \in \mathcal{A}(D\times\Sigma)$ of solutions of (2) with respect to metrics $g^\nu\in\mathfrak{m}$. We suppose that for some $2<p<3$ there is no local L^p-bound on the curvature near $\{0\}\times\Sigma$. By lemma 2.4 one then finds a subsequence (still denoted $(\Xi^\nu)_{\nu\in\mathbb{N}}$) and $D \ni (\bar{s}^\nu,\bar{t}^\nu) \to 0$ such that $\bar{R}^\nu := \|F_{\Xi^\nu}(\bar{s}^\nu,\bar{t}^\nu)\|_{L^2(\Sigma)} \to \infty$. We pick $\bar{\varepsilon}^\nu \to 0$ such that still $\bar{\varepsilon}^\nu\bar{R}^\nu \to \infty$. The Hofer trick, lemma 2.6, then yields $D \ni (s^\nu,t^\nu) \to 0$ and $\varepsilon^\nu \to 0$ such that $\|F_{\Xi^\nu}(s^\nu,t^\nu)\|_{L^2(\Sigma)} = R^\nu$ with $\varepsilon^\nu R^\nu \to \infty$ and

$$\bigl\| F_{\Xi^\nu}(s,t) \bigr\|_{L^2(\Sigma)} \leq 2R^\nu \qquad \forall (s,t) \in D_{2\varepsilon^\nu}(s^\nu,t^\nu).$$

Next, proposition 2.7 (ii) asserts that for all $\nu \geq \nu_0$

$$\bigl\| F_{A^\nu}(s,t) \bigr\|_{L^\infty(\Sigma)} \leq C\bigl(R^\nu\bigr)^2 \qquad \forall (s,t) \in D_{\varepsilon^\nu}(s^\nu,t^\nu).$$

Here and in the following C denotes any uniform constant. Now consider the functions $e^\nu = \|F_{\Xi^\nu}\|^2 : D \to [0,\infty)$. Use lemma 2.3 and the above bound to see that these satisfy on $D_{\varepsilon^\nu}(s^\nu,t^\nu)$

$$\Delta e^\nu \;\leq\; Ce^\nu - 20\langle\, F_{A^\nu} \,,\, [B^\nu_s \wedge B^\nu_s] \,\rangle_{L^2(\Sigma)} \;\leq\; C\bigl(1+(R^\nu)^2\bigr)e^\nu.$$

For the normal derivative we obtain from lemma 2.3 with a uniform constant A

$$-\tfrac{\partial}{\partial t}\big|_{t=0} e^\nu \;\leq\; C e^\nu - 4\int_\Sigma \langle\, \nabla_s B_s^\nu \wedge B_s^\nu \,\rangle \;\leq\; A\big(e^\nu + (e^\nu)^{\frac{3}{2}}\big).$$

Next, let $\mu > 0$ be the constant from the mean value inequality proposition 2.5. Now if $\nu \geq \nu_0$ and

$$\int_{D_{\varepsilon^\nu}(s^\nu,t^\nu)} e^\nu \leq \mu A^{-2} \tag{6}$$

then this proposition asserts that

$$(R^\nu)^2 \;=\; e^\nu(s^\nu,t^\nu) \;\leq\; C\big((R^\nu)^2 + (\varepsilon^\nu)^{-2} + 1\big)\int_{D_{\varepsilon^\nu}(s^\nu,t^\nu)} e^\nu .$$

From this it would follow that

$$\int_{D_{\varepsilon^\nu}(s^\nu,t^\nu)} e^\nu \;\geq\; C^{-1}\big(1 + (\varepsilon^\nu R^\nu)^{-2} + (R^\nu)^{-2}\big)^{-1}.$$

Hence for all $\nu \geq \nu_0$ we must either have (6) violated or this inequality holds true. Now for sufficiently large ν the right hand side is bounded below by $\frac{1}{2}C^{-1}$, thus in any case

$$\int_{D_{\varepsilon^\nu}(s^\nu,t^\nu)\times\Sigma} \big|F_{\Xi^\nu}\big|^2 \;>\; \min(\tfrac{1}{2}C^{-1}, \mu C^{-2}) \;=:\; \varepsilon_0 .$$

□

The proof of proposition 2.7 is based on the following boundary regularity result for anti-self-dual instantons on the half space with slicewise flat boundary conditions. These will arise from a local rescaling construction.

Here we use the coordinates (x,y,s,t) with $t \geq 0$ on $\mathbb{H}^4$, and as before we write connections $\Xi \in \mathcal{A}(\mathbb{H}^4)$ in the splitting $\Xi = A + \Phi \mathrm{d}s + \Psi \mathrm{d}t$. Note that under the assumptions of the following lemma (with any $p > 2$), the strong Uhlenbeck compactness for anti-self-dual connections (e.g. [W1, Thm E]) immediately implies the $\mathcal{C}^\infty$-convergence of a subsequence of connections (in a suitable gauge) in the interior, away from $\partial\mathbb{H}^4$. The slicewise flat boundary conditions are not quite enough to also obtain this convergence at the boundary, however we still obtain some partial regularity results for this boundary value problem. These provide the $\mathcal{C}^0$-convergence of the curvature component F_A, that vanishes on the boundary.

Lemma 2.8 *Let $p > \frac{8}{3}$ and let $D_1(0) \subset \mathbb{H}^4$ be the unit half ball of radius 1. Let g^ν be a sequence of metrics on $D_1(0)$ that converges to the Euclidean metric in the $\mathcal{C}^3$-norm. Let $\Xi^\nu \in \mathcal{A}^{1,p}(D_1(0))$ be a sequence of anti-self-dual connections with respect to the metrics g^ν and that satisfy flat boundary conditions, $F_{A^\nu}|_{t=0} \equiv 0$. Suppose that*

$$\lim_{\nu\to\infty} \big\|F_{\Xi^\nu}\big\|_{L^p(D_1(0))} = 0.$$

Then there exists a subsequence such that

$$\lim_{\nu\to\infty} \big\|F_{A^\nu}\big\|_{L^\infty(D_{\frac{1}{2}}(0))} = 0.$$

Proof: Let $U \subset \mathbb{H}^4$ be a compact submanifold with smooth boundary obtained from $D_{\frac{3}{4}}(0)$ by 'rounding off the edge' at $\partial\mathbb{H}^4$, so $D_{\frac{1}{2}}(0) \subset U \subset D_1(0)$. (More precisely, Uhlenbeck's gauge theorem below requires that the domain U is diffeomorphic to a ball; to obtain uniform constants, it should moreover be starlike w.r.t. 0.) Let a sequence of connections Ξ^ν as above be given. For sufficiently large ν the metrics g^ν on U are all sufficiently $\mathcal{C}^2$-close to the Euclidean metric so that the Uhlenbeck gauge for $\Xi^\nu|_U$ exists with uniform constants: The energy $\int_U |F_{\Xi^\nu}|^2$ becomes arbitrarily small for large ν, so by [U2, Thm 1.3] or [W1, Thm 6.3] these connections can be put into a gauge (again denoted by $\Xi^\nu \in \mathcal{A}^{1,p}(U)$) such that $\mathrm{d}^*\Xi^\nu = 0$ and $*\Xi^\nu|_{\partial U} = 0$. This gauge also gives a uniform bound $\|\Xi^\nu\|_{W^{1,p}(U)} \leq C_{Uh}\|F_{\Xi^\nu}\|_{L^p(U)} \leq C$.

We now have to follow through the higher regularity arguments in the proof of [W3, Thm 2.6] to find a uniform bound on F_{A^ν} in the Hölder norm $\mathcal{C}^{0,\lambda}(D_{\frac{1}{2}}(0))$ for some $\lambda > 0$. This will finish the proof since the embedding $\mathcal{C}^{0,\lambda} \hookrightarrow \mathcal{C}^0$ is compact, so this would imply $\mathcal{C}^0$-convergence for a subsequence. The limit can only be 0 since that was already the L^p-limit on $D_1(0)$.

Firstly note that the metrics on U are all $\mathcal{C}^3$-close, so [W3, Thm 2.6] allows for uniform estimates up to the third derivatives of the connections.[4] Since $2 < p < 3$ we are dealing with 'Boundary case II' in the proof of this theorem.

We choose $d > \frac{1}{2}$ such that $Q_d := [-d,d] \times [0,d] \times B_d \subset U$, where $B_d \subset \mathbb{R}^2$ is the Euclidean ball centered at 0. We moreover drop the superscript ν. Then the connections $\Xi = A + \Phi\mathrm{d}s + \Psi\mathrm{d}t$ with $A : [-d,d] \times [0,d] \to \Omega^1(B_d;\mathfrak{g})$ and $\Phi, \Psi : [-d,d] \times [0,d] \to \Omega^0(B_d;\mathfrak{g})$ solve the following boundary value problem analogous to [W3, (12)].[5] Here Q_d is equipped with the metric $\mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}$, and we shall write d, d* and ∇ for the families of operators on B_d with respect to the metrics $g_{s,t}$. Note that due to the localization we only retain the flat boundary condition.

$$\begin{cases} \mathrm{d}^*A = \partial_s\Phi + \partial_t\Psi, \\ *F_A = \partial_t\Phi - \partial_s\Psi + [\Psi,\Phi], \\ \partial_s A + *\partial_t A = \mathrm{d}_A\Phi + *\mathrm{d}_A\Psi, \\ \Psi(s,0) = 0 \quad \forall s \in [-d,d], \\ F_A(s,0) = 0 \quad \forall s \in [-d,d]. \end{cases} \tag{7}$$

Firstly, this combines to Laplace equations on Φ and Ψ (see (8) below) with a Dirichlet boundary condition for Ψ and an inhomogeneous Neumann condition for Φ,

$$\partial_t\Phi|_{t=0} = \partial_s\Psi - [\Psi,\Phi].$$

By e.g. [W3, Prop 2.7] this yields $W^{2,q}$-bounds for Φ and Ψ on Q_d with a slightly smaller $d > \frac{1}{2}$. Due to nonlinearities in the lower order terms, these bounds hold only for $q = \frac{4p}{8-p}$ (i.e. when $W^{1,p} \cdot L^p \hookrightarrow L^q$). However, we have assumed $p > \frac{8}{3}$ so that $q > 2$ and thus $W^{2,q}(Q_d)$ embeds into $\mathcal{C}^0(Q_d)$.

[4]The original theorem requires $\mathcal{C}^5$-bounds, but $\mathcal{C}^3$-bounds suffice when the metrics are already given in the appropriate coordinates (that otherwise are determined by the metrics).

[5]Note that B is replaced by A and we use the reference connection $A_0 = 0$.

Next, one has $W^{1,q}$-bounds on d^*A and $\mathrm{d}A$ as in [W3, (13)]. These lead to a bound on $\nabla A \in W^{1,q}(Q_d)$ (again for slightly smaller $d > \frac{1}{2}$), see [W3, Lemma 2.9]. In particular, A is bounded in $W^{1,q}([-d,d]\times[0,d], W^{1,q}(B_d))$, which embeds into $\mathcal{C}^0(Q_d)$. Thus we have obtained $\mathcal{C}^0$-bounds on the whole connection Ξ. Using these in the nonlinear terms and going through the previous two steps again yields bounds on $\Phi, \Psi \in W^{2,p}(Q_d)$ and $\nabla A \in W^{1,p}(Q_d)$ (with slightly smaller $d > \frac{1}{2}$). In order to obtain bounds for third derivatives of Φ and Ψ we calculate

$$\begin{aligned}\Delta\Phi &= \partial_s\bigl(\partial_t\Psi - \mathrm{d}^*A\bigr) + \partial_t\bigl([\Phi,\Psi] - \partial_s\Psi - *F_A\bigr) \\ &\quad + \mathrm{d}^*\bigl(\partial_s A + *\partial_t A + *\mathrm{d}_A\Psi - [A,\Phi]\bigr) \\ &= \partial_t\bigl([\Phi,\Psi] - *[A\wedge A]\bigr) - \mathrm{d}^*[A,\Phi] - *\mathrm{d}[A,\Psi] + \text{l.o.}\,.\end{aligned} \tag{8}$$

Here we have disregarded all lower order terms that arise from derivatives of the metric. From this one obtains an L^q-bound on $\Delta\nabla\Phi$. Indeed, the crucial terms are $*[A\wedge\nabla\partial_t A]$ and $*[\nabla A\wedge\partial_t A]$, where in both cases the first factor is $W^{1,p}$-bounded and the second factor is L^p-bounded. The analogous calculation also works for Ψ, so with the boundary conditions as before we obtain (for smaller $d > 0$) bounds on $\nabla\Phi, \nabla\Psi \in W^{2,q}(Q_d)$. In particular this gives bounds for Φ and Ψ in $W^{2,q}([-d,d]\times[0,d], W^{1,q}(B_d))$, and thus $*F_A = \partial_t\Phi - \partial_s\Psi + [\Psi,\Phi]$ is bounded in $W^{1,q}([-d,d]\times[0,d], W^{1,q}(B_d))$. Now finally, there is a continuous embedding of $W^{1,q}$ (on a 2-dimensional domain with values in any Banach space) into the Hölder space $\mathcal{C}^{0,2\lambda}$ with some $\lambda > 0$, so the above space embeds into $\mathcal{C}^{0,2\lambda}([-d,d]\times[0,d], \mathcal{C}^{0,2\lambda}(B_d))$, which in turn embeds continuously into $\mathcal{C}^{0,\lambda}(Q_d)$. Thus we obtain the claimed uniform bounds on $F_{A^\nu} \in \mathcal{C}^{0,\lambda}(Q_d)$. □

Proof of proposition 2.7: (i) and (ii) are proven by the same contradiction.

If (ii) was not true, then one would have a $\mathcal{C}^3$-compact set $\mathfrak{m}$ of metrics of normal type on $D\times\Sigma$ and $\Delta > 0$ with the following significance. For all $k\in\mathbb{N}$ there is a sequence $\Xi_k^\nu \in \mathcal{A}(D\times\Sigma)$ of solutions of (2) with respect to metrics $g_k^\nu \in \mathfrak{m}$, moreover $(\bar{s}_k^\nu, \bar{t}_k^\nu) \in D_{\frac{1}{2}r_0}$, $\varepsilon_k^\nu \to 0$, and $R_k^\nu \to \infty$ such that $\varepsilon_k^\nu R_k^\nu \geq \Delta > 0$ and

$$\bigl\|F_{\Xi_k^\nu}(s,t)\bigr\|_{L^2(\Sigma)} \leq R_k^\nu \qquad \forall (s,t)\in D_{2\varepsilon_k^\nu}(\bar{s}_k^\nu, \bar{t}_k^\nu).$$

But for every $k\in\mathbb{N}$ and $\nu_0\in\mathbb{N}$ there would exist $\nu\geq\nu_0$, $(s_k,t_k)\in D_{\varepsilon_k^\nu}(\bar{s}_k^\nu,\bar{t}_k^\nu)$, and $z_k\in\Sigma$ such that

$$\bigl|F_{A_k^\nu}(s_k,t_k,z_k)\bigr| > k\bigl(R_k^\nu\bigr)^2.$$

For each $k\in\mathbb{N}$ we choose ν_0 such that $\varepsilon_k^\nu \leq \frac{1}{k}$ and $R_k^\nu \geq k$ for all $\nu\geq\nu_0$. Then from a subsequence of a diagonal sequence one obtains solutions $\Xi^k\in\mathcal{A}(D\times\Sigma)$ of (2) w.r.t. a $\mathcal{C}^3$-convergent sequence of metrics $g^k\to g^\infty$; $\varepsilon^k\to 0$ and $R^k\to\infty$ such that $\varepsilon^k R^k \geq \Delta$; $(s^k,t^k)\to(s^\infty,t^\infty)\in D_{\frac{1}{2}r_0}$, $\Sigma\ni z^k\to z$, and $C^k\to\infty$ such that

$$\bigl|F_{A^k}(s^k,t^k,z^k)\bigr| \geq \bigl(C^k R^k\bigr)^2.$$

Since $D_{\varepsilon_k^\nu}(s_k,t_k)\subset D_{2\varepsilon_k^\nu}(\bar{s}_k^\nu,\bar{t}_k^\nu)$ one also obtains the bound

$$\bigl\|F_{\Xi^k}(s,t)\bigr\|_{L^2(\Sigma)} \leq R^k \qquad \forall (s,t)\in D_{\varepsilon^k}(s^k,t^k).$$

If (i) was not true, then one would find a sequence $\Xi^\nu \in \mathcal{A}(D\times\Sigma)$ solving (2), constants C, $0<\delta\leq\frac{1}{2}r_0$, and $(s^\nu,t^\nu,z^\nu)\in D_\delta(0)\times\Sigma$ such that $|F_{A^\nu}(s^\nu,t^\nu,z^\nu)|\to\infty$ but for all $\nu\in\mathbb{N}$

$$\sup_{(s,t)\in D_{2\delta}(0)} \bigl\|F_{\Xi^\nu}(s,t)\bigr\|_{L^2(\Sigma)} \leq C.$$

For a subsequence we can assume that $(s^\nu,t^\nu,z^\nu)\to(s^\infty,t^\infty,z^\infty)\in D_{\frac{1}{2}r_0}\times\Sigma$. We set $R^\nu = C^\nu := |F_{A^\nu}(s^\nu,t^\nu,z^\nu)|^{\frac{1}{4}}\to\infty$ and $\varepsilon^\nu := \min\bigl((R^\nu)^{-1},\delta\bigr)\to 0$. Then one has $C\leq R^\nu$, $\varepsilon^\nu R^\nu\geq 1 =: \Delta$, and $D_{\varepsilon^\nu}(s^\nu,t^\nu)\subset D_{2\delta}(0)$ for all sufficiently large $\nu\in\mathbb{N}$. That way, we have constructed the same sequences as for (ii), to which we shall find a contradiction:

- Solutions $\Xi^\nu\in\mathcal{A}(D\times\Sigma)$ of (2) with respect to $\mathcal{C}^3$-convergent metrics $g^\nu\to g^\infty$, constants $\varepsilon^\nu\to 0$, $R^\nu\to\infty$ with $\varepsilon^\nu R^\nu\geq\Delta>0$, and $C^\nu\to\infty$, and points $(s^\nu,t^\nu,z^\nu)\to(s^\infty,t^\infty,z^\infty)\in D_{\frac{1}{2}r_0}\times\Sigma$ such that

$$\sup_{(s,t)\in D_{\varepsilon^\nu}(s^\nu,t^\nu)} \bigl\|F_{\Xi^\nu}(s,t)\bigr\|_{L^2(\Sigma)}\leq R^\nu, \qquad \bigl|F_{A^\nu}(s^\nu,t^\nu,z^\nu)\bigr|\geq \bigl(C^\nu R^\nu\bigr)^2.$$

Firstly suppose that $\limsup_\nu t^\nu R^\nu\geq d>0$. In that case choose $d>0$ even smaller so $\delta\leq\Delta$, then $0<r^\nu := d(R^\nu)^{-1}\leq\varepsilon^\nu$ and $r^\nu\leq t^\nu$ for a suitable subsequence. Now the geodesic ball $B_{r^\nu}(s^\nu,t^\nu,z^\nu)$ with respect to g^ν is entirely contained in $D\times\Sigma$, and for sufficiently large ν it will be small enough to lie within a normal coordinate chart around $(s^\infty,t^\infty,z^\infty)$ for the metric g^∞. In this coordinate chart all metrics g^ν for large ν will be sufficiently $\mathcal{C}^1$-close to the Euclidean metric for proposition 2.2 to apply with uniform constants $\mu>0$ and C. Next, lemma 2.3 gives a uniform constant C_1 such that

$$\Delta\bigl|F_{\Xi^\nu}\bigr|^2 \;\leq\; C_1\bigl|F_{\Xi^\nu}\bigr|^2+8\bigl|F_{\Xi^\nu}\bigr|^3. \tag{9}$$

Let $\bar\mu := \frac{\mu}{64}$ and choose $d\leq\sqrt{\bar\mu/\pi}$ then

$$\int_{B_{r^\nu}(s^\nu,t^\nu,z^\nu)}\bigl|F_{\Xi^\nu}\bigr|^2 \;\leq\; \pi(r^\nu)^2\bigl(R^\nu\bigr)^2 \;\leq\; \bar\mu.$$

So proposition 2.2 implies

$$\bigl(C^\nu R^\nu\bigr)^4 \;\leq\; \bigl|F_{\Xi^\nu}(s^\nu,t^\nu,z^\nu)\bigr|^2 \;\leq\; C\bigl({C_1}^2+(r^\nu)^{-4}\bigr)\int_{B_{r^\nu}(s^\nu,t^\nu,z^\nu)}\bigl|F_{\Xi^\nu}\bigr|^2.$$

Putting in above estimate of the energy and $r^\nu R^\nu = d>0$ then leads to the contradiction

$$\bigl(C^\nu\bigr)^4 \;\leq\; C\bar\mu\bigl({C_1}^2\bigl(R^\nu\bigr)^{-4}+d^{-4}\bigr) \;\underset{\nu\to\infty}{\longrightarrow}\; C\bar\mu d^{-4} \;<\;\infty.$$

The second and crucial case is $t^\nu R^\nu\to 0$. We choose $\Delta\geq d>0$ and set $r^\nu := d(R^\nu)^{-1}\leq\varepsilon^\nu$ such that $\frac{1}{3}r^\nu\geq t^\nu$ for sufficiently large ν. Now for all $(s,t)\in D_{\frac{1}{3}r^\nu}(s^\nu,t^\nu)$ we have $t\leq t^\nu+\frac{1}{3}r^\nu\leq\frac{2}{3}r^\nu$, hence $B_t(s,t)\subset D_{\varepsilon^\nu}(s^\nu,t^\nu)$, and thus for all $z\in\Sigma$

$$\int_{B_t(s,t,z)} \bigl|F_{\Xi^\nu}\bigr|^2 \;\le\; \pi t^2 \bigl(R^\nu\bigr)^2 \;\le\; \tfrac{4}{9}\pi d^2 .$$

As in the first case one can choose ν sufficiently large such that for all $z \in \Sigma$ the above balls $B_t(s,t,z) \subset D_{\varepsilon^\nu}(s^\nu,t^\nu)\times\Sigma$ lie within a normal coordinate chart around (s^∞,t^∞,z) for the metric g^∞. Again, for large ν all metrics g^ν in these coordinates will be sufficiently $\mathcal{C}^1$-close to the Euclidean metric, so that (9) holds with a uniform constant C_1 and proposition 2.2 applies with uniform constants $\mu > 0$ and C. We choose $d > 0$ sufficiently small so that $\frac{4}{9}\pi d^2 \le \frac{\mu}{64}$, then proposition 2.2 implies that for all $(s,t,z) \in D_{\frac{1}{3}r^\nu}(s^\nu,t^\nu)\times\Sigma$

$$\bigl|F_{\Xi^\nu}(s,t,z)\bigr|^2 \;\le\; C\bigl({C_1}^2 + t^{-4}\bigr)\int_{B_t(s,t,z)} \bigl|F_{\Xi^\nu}\bigr|^2 \;\le\; C\pi\bigl(1+t^{-2}\bigr)\bigl(R^\nu\bigr)^2 .$$

Note here that $C_1 t \le C_1(r^\nu + t^\nu) \le 1$ for sufficiently large ν. With the above pointwise control of the curvature we can interpolate similar to lemma 2.4 to find for any fixed $2<p<3$ and for all $r \le \frac{1}{3}r^\nu$

$$\begin{aligned}
\int_{D_r(s^\nu,t^\nu)\times\Sigma} \bigl|F_{\Xi^\nu}\bigr|^p &\le \int_{D_r(s^\nu,t^\nu)} \bigl\|F_{\Xi^\nu}\bigr\|_{L^\infty(\Sigma)}^{p-2} \bigl\|F_{\Xi^\nu}\bigr\|_{L^2(\Sigma)}^2 \\
&\le C\bigl(R^\nu\bigr)^p \int_{D_r(s^\nu,t^\nu)} \bigl(1+t^{2-p}\bigr) \\
&\le C\bigl(R^\nu\bigr)^p \bigl(\pi r^2 + \tfrac{2r}{3-p}(t^\nu + r)^{3-p}\bigr) \;\le\; C\bigl(R^\nu\bigr)^p (t^\nu + r)^{4-p}
\end{aligned}$$

Here C denotes any uniform constant (depending on the choice of p). Next, recall that $|F_{A^\nu}(s^\nu,t^\nu,z^\nu)| \ge (C^\nu R^\nu)^2$ and $\varepsilon^\nu C^\nu R^\nu \ge \Delta C^\nu \to \infty$. So by the usual local rescaling we can define connections $\tilde\Xi^\nu$ on increasingly large 4-balls (intersected with half spaces) $B_{\varepsilon^\nu C^\nu R^\nu}(0) \cap \{t \ge -t^\nu C^\nu R^\nu\} \subset \mathbb{R}^4$. We use normal coordinates for g^∞ near $(s^\infty,t^\infty,z^\infty)$ and write $\mathbb{R}^4 = \{(s,t,z) \bigm| s,t\in\mathbb{R}, z\in\mathbb{R}^2\}$, then these connections are defined by

$$\tilde\Xi^\nu(s,t,z) := \Xi^\nu\bigl((s^\nu,t^\nu,z^\nu) + \tfrac{1}{C^\nu R^\nu}(s,t,z)\bigr).$$

They satisfy the boundary condition $F_{\tilde A^\nu}|_{t=-t^\nu C^\nu R^\nu} = 0$, and they are anti-self-dual with respect to the metrics $\tilde g^\nu(s,t,z) := g^\nu((s^\nu,t^\nu,z^\nu) + \frac{1}{C^\nu R^\nu}(s,t,z))$. Note that the coordinates were chosen such that on bounded domains the metric $\tilde g^\infty$ (rescaled by $C^\nu R^\nu$) converges to the Euclidean metric in any $\mathcal{C}^k$-norm. Thus for large ν the metrics $\tilde g^\nu$ become arbitrarily $\mathcal{C}^3$-close to the Euclidean metric.

Moreover, this construction is such that $|F_{\tilde A^\nu}(0)| \ge 1$ for all ν. On the other hand for all $\rho > 0$ we have $\rho(C^\nu R^\nu)^{-1} \le \frac{1}{3}r^\nu$ for sufficiently large ν and thus

$$\begin{aligned}
\bigl\|F_{\tilde\Xi^\nu}\bigr\|_{L^p(B_\rho(0)\cap\{t\ge -t^\nu C^\nu R^\nu\})}^p \;=\; &\bigl(C^\nu R^\nu\bigr)^{4-2p} \int_{D_{\rho(C^\nu R^\nu)^{-1}}(s^\nu,t^\nu,z^\nu)} \bigl|F_{\Xi^\nu}\bigr|^p \\
&\le C\bigl(C^\nu R^\nu\bigr)^{4-2p} \bigl(R^\nu\bigr)^p (t^\nu + \rho(C^\nu R^\nu)^{-1})^{4-p} \\
&\le C\bigl(C^\nu\bigr)^{4-2p} (t^\nu R^\nu + \rho(C^\nu)^{-1})^{4-p} \qquad (10) \\
&\underset{\nu\to\infty}{\longrightarrow}\; 0.
\end{aligned}$$

If $\limsup_\nu t^\nu C^\nu R^\nu > 0$, then we can choose a subsequence and $\rho > 0$ such that $F_{\tilde{\Xi}^\nu}$ is defined on $B_\rho(0)$ for all ν. Then the above estimate shows that $|F_{\tilde{\Xi}^\nu}| \to 0$ in the L^p-norm on $B_\rho(0)$ Due to the strong Uhlenbeck compactness for anti-self-dual connections (see e.g. [W1, Thm E]) one can find a subsequence and gauge transformations (which do not affect the norm of the curvature) such that this convergence is actually in the $\mathcal{C}^0$-topology. This contradicts $|F_{\tilde{A}^\nu}(0)| \geq 1$.

If $\tau^\nu := t^\nu C^\nu R^\nu \to 0$ then we need lemma 2.8 to obtain this contradiction. We shift the connections $\tilde{\Xi}^\nu$ and metrics $\tilde{g}^\nu$ by τ^ν in the t-direction so they are defined on $D_{\varepsilon^\nu C^\nu R^\nu}(0, \tau^\nu, 0)$. In particular, for sufficiently large ν, they are all defined on $D_1(0)$. Now the $\tilde{\Xi}^\nu$ satisfy flat boundary conditions at $t = 0$, and they are anti-self-dual with respect to the shifted metrics $\tilde{g}^\nu$. Since the shifts τ^ν converge to 0, we moreover preserve the $\mathcal{C}^3$-convergence of the metrics $\tilde{g}^\nu$ to the Euclidean metric. By this shift we have $|F_{\tilde{A}^\nu}(0, t^\nu, 0)| \geq 1$. Moreover, choose any $\frac{8}{3} < p < 3$, then we have $\left\|F_{\tilde{\Xi}^\nu}\right\|_{L^p(D_1(0))} \to 0$ since for ν sufficiently large $D_1(0) \subset D_2(0, \tau^\nu, 0)$, and the latter is the domain in (10) after the shifting. Now lemma 2.8 asserts that in fact $F_{\tilde{A}^\nu}$ converges to 0 in the $\mathcal{C}^0$-norm on $D_{\frac{1}{2}}(0)$. This however contradicts the fact that $|F_{\tilde{A}^\nu}(z^\nu)| \geq 1$ for $z^\nu = (0, t^\nu, 0) \to 0$. $\square$

3 Extension of connections in $\mathcal{L}_{\mathbf{Y}}$

This section is devoted to the proof of lemma 1.6, that is to extension constructions that relate connections in the Lagrangian $\mathcal{L}_Y$ on $\Sigma = \partial Y$ to flat connections on Y. Throughout Y is a handle body with boundary $\partial Y = \Sigma$ a Riemann surface of genus g. We moreover fix some $p > 2$. The Lagrangian $\mathcal{L}_Y \subset \mathcal{A}^{0,p}(\Sigma)$ as introduced in [W2, Lemma 4.6] is given by

$$\begin{aligned}\mathcal{L}_Y &= \mathrm{cl}_{L^p}\left\{A \in \mathcal{A}_{\mathrm{flat}}(\Sigma) \,\middle|\, \exists \tilde{A} \in \mathcal{A}_{\mathrm{flat}}(Y) : \tilde{A}|_\Sigma = A\right\} \\ &= \left\{u^*(A|_\Sigma) \,\middle|\, A \in \mathcal{A}_{\mathrm{flat}}(Y), u \in \mathcal{G}^{1,p}(\Sigma)\right\} \\ &= \left\{A \in \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma) \,\middle|\, \rho_z(A) \in \mathrm{Hom}(\pi_1(Y,z), \mathfrak{g}) \subset \mathrm{Hom}(\pi_1(\Sigma,z), \mathrm{G})\right\}.\end{aligned}$$

The space $\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ of weakly flat L^p-connections was introduced in [W2, Sec.3]. If we fix any $z \in \Sigma$, then every weakly flat connection is gauge equivalent to a smooth connection via a gauge transformation in the based gauge group

$$\mathcal{G}^{1,p}_z(\Sigma) = \left\{u \in \mathcal{G}^{1,p}(\Sigma) \,\middle|\, u(z) = \mathbb{1}\right\}.$$

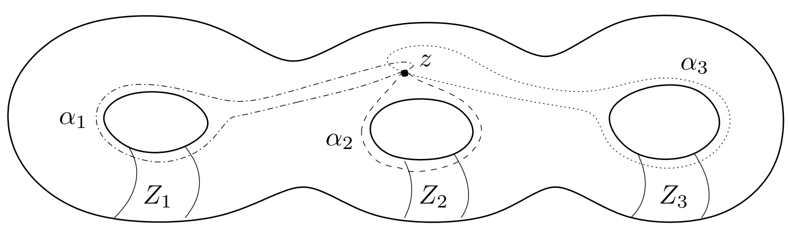
Thus the based holonomy ρ_z is welldefined on $\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ by first going to a smooth gauge and then calculating the holonomy along fixed generators of $\pi_1(\Sigma, z)$. We moreover recall from [W2] that $\mathcal{L}_Y$ is a Banach submanifold of $\mathcal{A}^{0,p}(\Sigma)$, and it is Lagrangian with respect to the symplectic form $\omega(\alpha,\beta) = \int_\Sigma \langle\, \alpha \wedge \beta \,\rangle$ in the sense that $\mathrm{T}_A\mathcal{L}_Y \subset L^p(\Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ is a maximal isotropic subspace for all $A \in \mathcal{L}_Y$. Finally, $\mathcal{L}_Y$ has the structure of a $\mathcal{G}^{1,p}_z(\Sigma)$-bundle over the g-fold product $\mathrm{G} \times \cdots \times \mathrm{G} \cong \mathrm{Hom}(\pi_1(Y,z), \mathrm{G})$,

$$\mathcal{G}^{1,p}_z(\Sigma) \;\hookrightarrow\; \mathcal{L}_Y \;\xrightarrow{\rho_z}\; \mathrm{G}^g.$$

We will fix a bundle atlas by specifying local sections over a finite cover of G^g. For that purpose we choose loops $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \subset \Sigma$ disjoint from z that represent the standard generators of $\pi_1(\Sigma)$ such that $\alpha_1, \ldots, \alpha_g$ generate $\pi_1(Y)$ and the only nonzero intersections are $\alpha_i \cap \beta_i$. One can then modify the α_i such that they run through z and coincide in a neighbourhood of z but still do not intersect the β_j for $j \neq i$.

The based holonomy $\rho_z : \mathcal{L}_Y \to \mathrm{G}^g \cong \mathrm{Hom}(\pi_1(Y, z), \mathrm{G})$ is now given by the g holonomies $\mathrm{hol}_{\alpha_i} : \mathcal{L}_Y \to \mathrm{G}$ for the paths α_i starting and ending at z.

Next, we choose spanning discs of the β_i that are pairwise disjoint and intersect ∂Y in β_i only. Their tubular neighbourhoods provide orientation preserving diffeomorphisms $\psi_i : [0,1] \times D \to Z_i$ (with $D \subset \mathbb{R}^2$ the unit disc) to disjoint neighbourhoods $Z_i \subset Y$ of the spanning discs. They can be chosen such that $\alpha_i \cap Z_j = \emptyset$ for $i \neq j$ and such that $\psi_i : [0,1] \times \{y\} \xrightarrow{\sim} \alpha_i \cap Z_i$ for some $y \in \partial D$. We then fix the induced orientation for the α_i.



Choose $\Delta > 0$ less than the injectivity radius of $\exp : \mathfrak{g} \to \mathrm{G}$ and fix a function $\tau \in \mathcal{C}^\infty([0,1],[0,1])$ with $\tau|_{[0,\frac{1}{4}]} \equiv 0$ and $\tau|_{[\frac{3}{4},1]} \equiv 1$. Now given any fixed $\Theta^0 = (\theta^0_1, \ldots, \theta^0_g) \in \mathrm{G}^g$ we choose smooth paths $\gamma^0_i : [0,1] \to \mathrm{G}$ with $\gamma^0_i|_{[0,\frac{1}{4}]} \equiv \mathbb{1}$ and $\gamma^0_i|_{[\frac{3}{4},1]} \equiv \theta_i^{-1}$. Let $B_\Delta(\Theta^0) \subset \mathrm{G}^g$ be the closed exponential ball around Θ^0. Then for every $\Theta = (\theta_i) \in B_\Delta(\Theta^0)$ we have local gauge transformations $v_i \in \mathcal{G}(Z_i)$ given by $v_i(\psi_i(t,x)) = (\gamma^0_i(t) \exp(\tau(t)\xi_i))^{-1}$, where $\xi_i = \exp^{-1}((\theta^0_i)^{-1}\theta_i) \in B_\Delta(0) \subset \mathfrak{g}$. Note that $v_i \equiv \mathbb{1}$ and $v_i \equiv \theta_i^{-1}$ near the two boundary components of $Z_i \subset Y$. These local gauge transformations can be used to define a local section of $\mathcal{L}_Y$, that is a smooth map $\Xi : B_\Delta(\Theta^0) \to \mathcal{A}_{\mathrm{flat}}(Y)$ such that $\rho_z(\Xi(\Theta)|_\Sigma) = \Theta$,

$$\Xi(\Theta) \;=\; \begin{cases} v_i^{-1}\mathrm{d}v_i & ; \text{on } Z_i, \\ 0 & ; \text{on } Y \setminus \bigcup_{i=1}^g Z_i. \end{cases} \tag{11}$$

We now fix $\Theta^0_j \in \mathrm{G}^g$ for $j = 1, \ldots, N$ such that the domains $B_{\frac{1}{2}\Delta}(\Theta^0_j)$ already cover all of G^g. This gives rise to a bundle atlas for $\mathcal{L}_Y$ given by the charts

$$\begin{array}{ccc} \mathcal{G}^{1,p}_z(\Sigma) \times B_\Delta(\Theta^0_j) & \longrightarrow & \mathcal{L}_Y \\ (u, \Theta) & \longmapsto & u^*\Xi_j(\Theta)|_\Sigma. \end{array} \tag{12}$$

Next, we can find tubular neighbourhoods $\tilde{\alpha}_i : [-1,1] \times [0,1] \hookrightarrow \Sigma$ of the loops $\alpha_i = \tilde{\alpha}_i(0,\cdot)$ that again coincide near z for all $i = 1, \ldots, g$. Then these are a family of loops based at $\tilde{\alpha}_i(\tau, 0) = \tilde{\alpha}_i(\tau, 1) = z(\tau)$ for some i-independent

smooth path $z : [-1,1] \to \Sigma \setminus \bigcup Z_i$. As before, the intersection $\tilde\alpha_i(\tau,\cdot) \cap Z_j$ will be empty for $i \neq j$, and for $i = j$ it is $\psi_i([0,1] \times \{y(\tau)\}$ for some $y(\tau) \in \partial D$.

Note that for the special connections $\Xi(\Theta) \in \mathcal{A}_{\rm flat}(Y)$ as in (11) the holonomies $\mathrm{hol}_{\tilde\alpha_i(\tau)}(\Xi(\Theta)) = \mathrm{hol}_{\alpha_i}(\Xi(\Theta))$ are independent of $\tau \in [-1,1]$. For other connections, the variation of the paths $\tilde\alpha_i(\tau,\cdot)$ along $\tau \in [-1,1]$ allows to control the holonomy by the connections in the L^1-topology.

Lemma 3.1 *There exists a constant C such that the following holds.*

(i) For all smooth paths $A : (-\varepsilon,\varepsilon) \to \mathcal{A}(\Sigma)$ there exists $\tau \in [-1,1]$ such that with $\theta = \mathrm{hol}_{\tilde\alpha_i(\tau)}(A(0))$ for all $i = 1,\dots,g$

$$\bigl|\partial_s|_{s=0}\mathrm{hol}_{\tilde\alpha_i(\tau)}(A(s))\bigr|_{\mathrm{T}_\theta \mathrm{G}} \;\leq\; C\|\partial_s A(0)\|_{L^1(\Sigma)}.$$

(ii) For all $A_0, A_1 \in \mathcal{A}(\Sigma)$ there exists $\tau \in [-1,1]$ such that for all $i = 1,\dots,g$

$$\mathrm{dist}_{\mathrm{G}}\bigl(\mathrm{hol}_{\tilde\alpha_i(\tau)}(A_0)\,,\,\mathrm{hol}_{\tilde\alpha_i(\tau)}(A_1)\bigr) \;\leq\; C\|A_0 - A_1\|_{L^1(\Sigma)}.$$

Proof: Starting with the proof of (ii) we recall that for every $i = 1,\dots,g$ and all $\tau \in [-1,1]$ the holonomies $\mathrm{hol}_{\tilde\alpha_i(\tau)}(A_j) = u_j(1) \in \mathrm{G}$ for $j = 0,1$ are given by the solutions $u_j : [0,1] \to \mathrm{G}$ of

$$\dot u_j u_j^{-1} = -\tilde\alpha_i(\tau)^* A_j \qquad \text{with} \qquad u_j(0) = \mathbb{1}.$$

Note that for fixed $i = 1,\dots,g$ and $\tau \in [-1,1]$

$$\partial_t\bigl(u_0^{-1}u_1\bigr) \;=\; -u_0^{-1}\dot u_0 u_0^{-1}u_1 + u_0^{-1}\dot u_1 \;=\; u_0^{-1}\tilde\alpha_i(\tau)^*(A_0 - A_1)\,u_1.$$

Hence

$$\begin{aligned}
\mathrm{dist}_{\mathrm{G}}\bigl(\mathrm{hol}_{\tilde\alpha_i(\tau)}(A_0)\,,\,\mathrm{hol}_{\tilde\alpha_i(\tau)}(A_1)\bigr) &= \mathrm{dist}_{\mathrm{G}}\bigl(\mathbb{1},\, u_0(1)^{-1}u_1(1)\bigr) \\
&\leq \int_0^1 \bigl|\partial_t\bigl(u_0(t)^{-1}u_1(t)\bigr)\bigr|\,\mathrm{d}t \\
&\leq \int_0^1 \bigl|\tilde\alpha_i(\tau)^*(A_0 - A_1)\bigr|\,\mathrm{d}t.
\end{aligned}$$

Next, for every $i = 1,\dots,g$ there exists a set $V_i \subset [-1,1]$ of measure $|V_i| \geq 2 - \frac{1}{g}$ such that for all $\tau \in V_i$

$$\begin{aligned}
\int_0^1 \bigl|\tilde\alpha_i(\tau)^*(A_0 - A_1)\bigr|\,\mathrm{d}t \;&\leq\; g\int_{-1}^1\int_0^1 \bigl|\tilde\alpha_i(\tau)^*(A_0 - A_1)\bigr|\,\mathrm{d}t\,\mathrm{d}\tau \\
&\leq\; C\int_{\tilde\alpha_i} |A_0 - A_1| \;\leq\; C\|A_0 - A_1\|_{L^1(\Sigma)}.
\end{aligned}$$

Here the constant C only depends on the embeddings $\tilde\alpha_i$. Now the claim (ii) is true for all $\tau \in \bigcap_{i=1}^g V_i$, which is nonempty. In case (i) we similarly find $\tau \in [-1,1]$ such that

$$\int_0^1 \bigl|\tilde\alpha_i(\tau)^*(\partial_s A(0))\bigr|\,\mathrm{d}t \;\leq\; C\|\partial_s A(0)\|_{L^1(\Sigma)}.$$

Now with $\theta = \mathrm{hol}_{\tilde\alpha_i(\tau)}(A(0))$ we obtain as above

$$
\begin{aligned}
\left|\partial_s|_{s=0}\mathrm{hol}_{\tilde\alpha_i(\tau)}(A(s))\right|_{\mathrm{T}_\theta \mathrm{G}} &= \lim_{s\to 0} |s|^{-1}\mathrm{dist}_{\mathrm{G}}\bigl(\mathrm{hol}_{\tilde\alpha_i(\tau)}(A(0))\,,\, \mathrm{hol}_{\tilde\alpha_i(\tau)}(A(s))\bigr) \\
&\le \lim_{s\to 0}\int_0^1 \left|\tilde\alpha_i(\tau)^*\left(\frac{A(0)-A(s)}{s}\right)\right| \mathrm{d}t \\
&= \int_0^1 \left|\tilde\alpha_i(\tau)^*\partial_s A(0)\right| \mathrm{d}t. \qquad \square
\end{aligned}
$$

Now consider the extension problems in lemma 1.6. Given connections in $\mathcal{L}_Y$, the above lemma provides a control of the holonomies based at some point $z(\tau)$. This point can vary in a neighbourhood of $z \in \Sigma$. However, for any such basepoint, the sections (11) will provide flat connections on Y with the holonomy of the given connections on Σ. So on $\partial Y = \Sigma$, these connections only differ by a gauge transformation. Thus we require the following extension construction for gauge transformations. Here and in the following d^Σ_Ξ for $\Xi \in \mathcal{A}(Y)$ denotes the exterior derivative on $\mathcal{A}(\Sigma)$ associated with the connection $\Xi|_\Sigma$.

Lemma 3.2 *There is a constant C such that the following holds for any connection $\Xi = \Xi_j(\Theta) \in \mathcal{A}_{\mathrm{flat}}(Y)$, $\Theta \in B_\Delta(\Theta^0_j)$ in the finitely many sections (11).*

(i) For all $\xi \in \mathcal{C}^\infty(\Sigma,\mathfrak{g})$ there exists $\tilde\xi \in \mathcal{C}^\infty(Y,\mathfrak{g})$ such that $\tilde\xi|_{\partial Y} = \xi$ and

$$
\bigl\|\mathrm{d}_\Xi\tilde\xi\bigr\|_{L^3(Y)} \;\le\; C\|\mathrm{d}^\Sigma_\Xi\xi\|_{L^2(\Sigma)}.
$$

(ii) For all $u \in \mathcal{G}(\Sigma)$ there exists $\tilde u \in \mathcal{G}(Y)$ such that $\tilde u|_{\partial Y} = u$ and

$$
\bigl\|\tilde u^*\Xi - \Xi\bigr\|_{L^3(Y)} \;\le\; C\|u^*\Xi|_\Sigma - \Xi|_\Sigma\|_{L^2(\Sigma)}.
$$

For (ii) note that a smooth map $\Sigma \to \mathrm{G}$ can always be extended to $Y \to \mathrm{G}$ since by assumption $\pi_1(\mathrm{G}) = 0$ (so extensions to discs in Y with boundary in Σ exist), and for general Lie groups $\pi_2(\mathrm{G}) = 0$ (so these extensions can be matched up). We will moreover use the following quantitative result with $N = \mathrm{G}$ and thus $\ell = 2$, where the Sobolev spaces of maps into $N \subset \mathbb{R}^k$ are understood as

$$
W^{1,q}(\Omega, N) = \bigl\{ u \in W^{1,q}(\Omega,\mathbb{R}^k) \bigm| \forall' x \in \Omega : u(x) \in N \bigr\}.
$$

Theorem 3.3 [HnL] *Let $N \subset \mathbb{R}^k$ be a smooth connected compact Riemannian manifold with $\pi_i(N) = 0$ for all $i = 1, \ldots, \ell$. Then the following holds for all $1 < q < \ell + 2$. Let $\Omega \subset \mathbb{R}^m$ be an open, bounded domain with piecewise smooth boundary. Then there exists a constant C such that for any $u \in W^{1-\frac{1}{q},q}(\partial\Omega, N)$ there exists $\tilde u \in W^{1,q}(\Omega, N)$ with $\tilde u|_{\partial\Omega} = u$ such that*

$$
\|\mathrm{d}\tilde u\|_{L^q(\Omega)} \le C\|u\|_{W^{1-\frac{1}{q},q}(\partial\Omega)}.
$$

In particular, if Ω is simply connected and if we fix $1 < \bar q \le q$ and $1 < \bar p \le p$ such that $p \ge \frac{m-1}{m}q$, $\bar p \ge \frac{m-1}{m}\bar q$, then there is a constant C such that for any $u \in W^{1,p}(\partial\Omega, N)$ there exists $\tilde u \in W^{1,q}(\Omega, N)$ with $\tilde u|_{\partial\Omega} = u$ such that

$$
\|\mathrm{d}\tilde u\|_{L^{\bar q}(\Omega)} \le C\|\mathrm{d}u\|_{L^{\bar p}(\partial\Omega)}.
$$

The first part is [HnL, Thm 2.1], and the second part is an easy consequence: One has $W^{1,p}(\partial\Omega) \hookrightarrow W^{1-\frac{1}{q},q}(\partial\Omega)$ and the trace $W^{2,\bar{q}^*}(\Omega) \hookrightarrow W^{1,\bar{p}^*}(\partial\Omega)$ by e.g. [A, Thms 7.8, 5.22]. If Ω is simply connected, then the operator $(\mathrm{d}, \mathrm{d}^*, \cdot|_{\partial\Omega})$ is injective, so as in the proof of lemma 3.2 (i) one finds for all $\alpha \in \Omega^1(\Omega)$

$$\|\alpha\|_{L^{\bar{q}}(\Omega)} \;\leq\; C\bigl(\|\mathrm{d}\alpha\|_{(W^{1,\bar{q}^*}(\Omega))^*} + \|\mathrm{d}^*\alpha\|_{(W^{1,\bar{q}^*}(\Omega))^*} + \|\alpha|_{\partial\Omega}\|_{L^{\bar{p}}(\partial\Omega)}\bigr).$$

The proof in [HnL] uses the solution $v \in W^{1,q}(\Omega, \mathbb{R}^k)$ of $\mathrm{d}^*\mathrm{d}v = 0$ with $v|_{\partial\Omega} = u$, for which in this case $\|\mathrm{d}v\|_{L^{\bar{q}}(\Omega)} \leq C\|\mathrm{d}u\|_{L^{\bar{p}}(\partial\Omega)}$. Variation of a 'centre' of a retraction to N then gives $\tilde{u} \in W^{1,q}(\Omega, N)$ with $\|\mathrm{d}\tilde{u}\|_{L^q(\Omega)} \leq C\|\mathrm{d}v\|_{L^q(\Omega)}$. This centre $a \in \mathbb{R}^k$ can be found to simultaneously yield the same estimate for $\bar{q}$.

Proof of lemma 3.2:
For (i) we determine $\tilde{\xi} \in \mathcal{C}^\infty(Y, \mathfrak{g})$ by solving the Dirichlet problem

$$\mathrm{d}_\Xi^*\mathrm{d}_\Xi\tilde{\xi} = 0, \qquad\qquad \tilde{\xi}|_{\partial Y} = \xi.$$

The operator $(\mathrm{d}_\Xi^*\mathrm{d}_\Xi, \cdot|_{\partial Y})$ on $W^{2,2}(Y, \mathfrak{g})$ is a compact perturbation of the standard Dirichlet operator $(\Delta, \cdot|_{\partial Y})$, so it is a Fredholm operator of index 0. It is surjective since its kernel equals $\ker(\mathrm{d}_\Xi, \cdot|_{\partial Y}) = \{0\}$, where a solution of $\mathrm{d}_\Xi\eta = 0$ is uniquely determined by its value at any one point via integration along paths. For $\xi \in \mathcal{C}^\infty(\Sigma, \mathfrak{g})$ the smoothness of the solution $\tilde{\xi}$ follows from elliptic regularity.

The estimate for $\mathrm{d}_\Xi\tilde{\xi}$ will be provided by the following Hodge type estimate: There exists a constant C, independent of Ξ, such that for all $\alpha \in \Omega^1(Y, \mathfrak{g})$.

$$\|\alpha\|_{L^3(Y)} \;\leq\; C\bigl(\|\mathrm{d}_\Xi\alpha\|_{L^{\frac{3}{2}}(Y)} + \|\mathrm{d}_\Xi^*\alpha\|_{L^{\frac{3}{2}}(Y)} + \|\alpha|_{\partial Y}\|_{L^2(\Sigma)}\bigr). \tag{13}$$

If we put in $\alpha = \mathrm{d}_\Xi\tilde{\xi}$, then $\mathrm{d}_\Xi\mathrm{d}_\Xi\tilde{\xi} = 0$ since Ξ is flat and $\mathrm{d}_\Xi^*\mathrm{d}_\Xi\tilde{\xi} = 0$ by construction. So it remains to establish (13). If we consider the normal and tangential components of the 1-forms on Y separately, then this estimate deals with the operator $\mathrm{d}_\Xi^*\mathrm{d}_\Xi$ with Dirichlet boundary conditions for the tangential components. From $\mathrm{d}_\Xi^*\alpha$ one also has a Neumann boundary condition for the normal component in terms of the tangential components. So a combination of Dirichlet estimates for the tangential components and a Neumann estimate for the normal component will imply (13).

More precisely, one can use [W1, Thm 5.3] to obtain $W^{1,\frac{3}{2}}$-estimates for $\alpha(X)$, where $X \in \Gamma(\mathrm{T}Y)$ is either tangential to ∂Y (in which case one uses test functions $\phi \in \mathcal{C}^\infty(Y, \mathfrak{g})$ with $\phi|_{\partial Y} = 0$), or X is normal to ∂Y (and one uses test functions with $\frac{\partial}{\partial\nu}\phi|_{\partial Y} = 0$). In both cases one then has the following estimates, where the constant C depends on Ξ. Firstly, the boundary term vanishes in

$$\begin{aligned}\left|\int_Y \langle\, \alpha \,,\, \mathrm{d}\mathcal{L}_X\phi \,\rangle\right| &= \left|\int_Y \langle\, \mathrm{d}_\Xi^*\alpha \,,\, \mathcal{L}_X\phi \,\rangle + \int_Y \langle\, *[\Xi \wedge *\alpha] \,,\, \mathcal{L}_X\phi \,\rangle + \int_{\partial Y} \langle\, *\alpha \,,\, \mathcal{L}_X\phi \,\rangle\right| \\ &\leq C\bigl(\|\mathrm{d}_\Xi^*\alpha\|_{L^{\frac{3}{2}}(Y)} + \|\alpha\|_{(W^{1,\frac{3}{2}}(Y))^*}\bigr)\|\phi\|_{W^{2,\frac{3}{2}}(Y)}.\end{aligned}$$

This also uses the Sobolev inequality $\|\phi\|_{W^{1,3}(Y)} \leq C\|\phi\|_{W^{2,\frac{3}{2}}(Y)}$. Secondly, one can use the Sobolev embedding $W^{2,\frac{3}{2}}(Y) \hookrightarrow W^{1,2}(\partial Y)$ (see [A, 5.22]) to obtain

$$\begin{aligned}
&\left| \int_Y \langle\, \alpha\,,\, \mathrm{d}^*(i_X g \wedge \mathrm{d}\phi)\,\rangle \right| \\
&= \left| \int_Y \langle\, \mathrm{d}_\Xi \alpha\,,\, i_X g \wedge \mathrm{d}\phi\,\rangle - \int_Y \langle\, [\Xi \wedge \alpha]\,,\, i_X g \wedge \mathrm{d}\phi\,\rangle - \int_{\partial Y} \langle\, \alpha \wedge *(i_X g \wedge \mathrm{d}\phi)\,\rangle \right| \\
&\leq C\big(\|\mathrm{d}_\Xi \alpha\|_{L^{\frac{3}{2}}(Y)} + \|\alpha\|_{(W^{1,\frac{3}{2}}(Y))^*} + \|\alpha|_{\partial Y}\|_{L^2(\Sigma)}\big)\|\phi\|_{W^{2,\frac{3}{2}}(Y)}.
\end{aligned}$$

These two estimates can be considered as weak Laplace equations on $\alpha(X)$ with inhomogenous Dirichlet or Neumann boundary conditions respectively. The according estimates sum up to

$$\|\alpha\|_{L^3(Y)} \;\leq\; C\big(\|\mathrm{d}_\Xi \alpha\|_{L^{\frac{3}{2}}(Y)} + \|\mathrm{d}_\Xi^* \alpha\|_{L^{\frac{3}{2}}(Y)} + \|\alpha|_{\partial Y}\|_{L^2(\Sigma)} + \|\alpha\|_{(W^{1,\frac{3}{2}}(Y))^*}\big).$$

Finally, the last term can be dropped since the embedding $L^3(Y) \hookrightarrow (W^{1,\frac{3}{2}}(Y))^*$ is the dual of a compact Sobolev embedding, and the operator $(\mathrm{d}_\Xi, \mathrm{d}_\Xi^*, \cdot|_{\partial Y})$ is injective. To see the latter consider an element $\alpha \in \Omega^1(Y, \mathfrak{g})$ of the kernel. We can write it as $\alpha = \mathrm{d}_\Xi \eta$ for some $\eta \in \mathcal{C}^\infty(Y, \mathfrak{g})$ with $\eta|_{\partial Y} = 0$. [6] Then $\mathrm{d}_\Xi^* \mathrm{d}_\Xi \eta = 0$ with $\eta|_{\partial Y} = 0$ implies $\alpha = \mathrm{d}_\Xi \eta = 0$ by partial integration. Thus (13) holds for every $\Xi \in \mathcal{A}_{\mathrm{flat}}(Y)$.

The constant in (13) depends continuously on Ξ with respect to the L^∞-norm. It can be chosen uniform since we only consider smooth connections Ξ that are parametrized by a finite number of compact sets $B_\Delta(\Theta_j^0) \subset \mathrm{G}^g$.

In (ii) we need to extend $u : \partial Y \to \mathrm{G}$ to $\tilde{u} : Y \to \mathrm{G}$. Our construction will make use of theorem 3.3, where we fix an embedding $\mathrm{G} \subset \mathbb{R}^k$ and some $2 < p < \frac{8}{3}$. We recall the diffeomorphisms $\psi_i : [0,1] \times D \to Z_i \subset Y$ and denote $D(\tau) := \psi_i(\tau, D) \subset Y$ with the orientation induced by ψ_i. By construction the connection Ξ vanishes over $D(\tau)$ for all $\tau \in [\frac{3}{4}, 1]$. So given any $u \in \mathcal{G}(\Sigma)$ we find $\tau_i \in [\frac{3}{4}, 1]$ for every $i = 1, \ldots, g$ such that

$$\int_{\partial D} |\psi_i(\tau_i)^* \mathrm{d}u|^2 \;\leq\; 4 \int_{[\frac{3}{4},1] \times \partial D} |\psi_i^* \mathrm{d}u|^2 \;=\; 4 \int_{[\frac{3}{4},1] \times \partial D} \big|\psi_i^*\big(u^*\Xi - \Xi\big)\big|^2.$$

Since the ψ_i are fixed we then have with a uniform constant C for all $i = 1, \ldots, g$

$$\|\mathrm{d}u\|_{L^2(\partial D(\tau_i))} \;\leq\; C\|u^*\Xi|_\Sigma - \Xi|_\Sigma\|_{L^2(\Sigma)}.$$

Now theorem 3.3 on $\Omega = D(\tau_i) \subset \mathbb{R}^2$ with $q = p > 2$ as fixed and $\bar{q} = \bar{p} = 2$ gives $\tilde{u}_i \in W^{1,p}(D(\tau_i), \mathrm{G})$ with $\tilde{u}_i|_{\partial D(\tau_i)} = u|_{\partial D(\tau_i)}$ and

$$\|\mathrm{d}\tilde{u}_i\|_{L^2(D(\tau_i))} \;\leq\; C\|u^*\Xi|_\Sigma - \Xi|_\Sigma\|_{L^2(\Sigma)}.$$

[6] Since $F_\Xi = 0$ and $\mathrm{d}_\Xi \alpha = 0$ this is true on simply connected subsets of Y. We can moreover prescribe $\eta|_{\partial Y} = 0$ since $\alpha|_{\partial Y} = 0$. Now Y can be covered with simply connected domains whose intersections are connected and meet ∂Y. (The 1-skeleton of Y can be pushed to ∂Y.) So if η and η' are each determined on one of these domains, then they have to match up on the intersection. This is since $\mathrm{d}_\Xi(\eta - \eta') = 0$ with $\eta = \eta'$ at one point only has the trivial solution $\eta = \eta'$.

Next, fix an embedding $Y \subset \mathbb{R}^3$ and cut Y open to obtain the simply connected open manifold $Y_{\bar\tau} = \mathrm{int}(Y) \setminus \bigcup_{i=1}^g D(\tau_i)$. For any choice of $\bar\tau = (\tau_i) \in [\frac{3}{4}, 1]^g$ this is diffeomorphic to the standard domain $\mathrm{int}(Y) \setminus \bigcup Z_i \subset \mathbb{R}^3$ with a uniform bound on every given derivative. Thus we can apply theorem 3.3 with a uniform constant to all these domains. Their piecewise smooth boundary then is

$$\partial Y_{\bar\tau} = \Sigma_{\bar\tau} \cup \bigcup_{i=1}^g D(\tau_i) \cup \bigcup_{i=1}^g \bar D(\tau_i) \qquad \text{with} \qquad \Sigma_{\bar\tau} = \Sigma \setminus \bigcup_{i=1}^g \partial D(\tau_i).$$

Here $D(\tau_i)$ is the boundary component attached to $\psi_i([0,\tau_i)\times D) \subset Y_{\bar\tau}$, whereas $\bar D(\tau_i)$ with the reversed orientation is attached to $Y_{\bar\tau} \setminus \psi_i([0,\tau_i)\times D)$. Now recall that $\Xi|_{Z_i} = v_i^{-1}\mathrm{d}v_i$, where v_i is smooth on $Z_i = \psi_i([0,1]\times D)$ and satisfies $v_i|_{\psi_i([0,\frac12]\times D)} \equiv \mathbb{1}$ and $v_i|_{D(\tau_i)} \equiv \theta_i^{-1} \in \mathrm{G}$. So we can write $\Xi|_{Y_{\bar\tau}} = v^{-1}\mathrm{d}v$, where $v \in \mathcal{C}^\infty(Y_{\bar\tau}, \mathrm{G})$ is given by $v = v_i$ on $\psi_i([0,\tau_i)\times D)$ and $v \equiv \mathbb{1}$ on the complement. With this we define

$$w := \begin{cases} v\,u\,v^{-1} & ;\text{on } \Sigma_{\bar\tau} \\ \tilde u_i & ;\text{on } \bar D(\tau_i) \\ \theta_i^{-1}\tilde u_i\,\theta_i & ;\text{on } D(\tau_i) \end{cases} \qquad \in\; W^{1,p}(Y_{\bar\tau}, \mathrm{G}).$$

This gauge transformation is chosen such that on $\Sigma_{\bar\tau}$

$$w^{-1}\mathrm{d}w \;=\; v\,u^{-1}v^{-1}\mathrm{d}v\,u\,v^{-1} + v\,u^{-1}\mathrm{d}u\,v^{-1} - v\,v^{-1}\mathrm{d}v\,v^{-1} \;=\; v(u^*\Xi - \Xi)v^{-1}.$$

So we can apply theorem 3.3 on $Y_{\bar\tau} \hookrightarrow \mathbb{R}^3$ with $q = \frac{3}{2}p$, $\bar p = 3$, and $\bar q = 2$ to obtain $\tilde w \in W^{1,\frac32 p}(Y_{\bar\tau}, \mathrm{G})$ such that $\tilde w|_{\partial Y_{\bar\tau}} = w$ and

$$\begin{aligned} \|\mathrm{d}\tilde w\|_{L^3(Y_{\bar\tau})} &\le C\|\mathrm{d}w\|_{L^2(\partial Y_{\bar\tau})} \\ &\le C\big(\|u^*\Xi - \Xi\|_{L^2(\Sigma_{\bar\tau})} + \|\mathrm{d}\tilde u_i\|_{L^2(D(\tau_i))} + \|\theta_i^{-1}\mathrm{d}\tilde u_i\theta_i\|_{L^2(D(\tau_i))}\big) \\ &\le C\|u^*\Xi|_\Sigma - \Xi|_\Sigma\|_{L^2(\Sigma)}. \end{aligned}$$

Now $\tilde u := v^{-1}\tilde w\,v \in W^{1,\frac32}(Y_{\bar\tau}, \mathrm{G})$ satisfies $\tilde u|_{\Sigma_{\bar\tau}} = u|_{\Sigma_{\bar\tau}}$ and $\tilde u|_{D(\tau_i)} = \tilde u_i = \tilde u_{\bar D(\tau_i)}$, so it matches up to $\tilde u \in W^{1,\frac32}(Y, \mathrm{G})$. Also,

$$\big(\tilde u^*\Xi - \Xi\big)\big|_{Y_{\bar\tau}} \;=\; v^*\tilde w^*(v^{-1})^*v^{-1}\mathrm{d}v - v^{-1}\mathrm{d}v \;=\; v^{-1}\tilde w^{-1}\mathrm{d}\tilde w\,v,$$

and hence

$$\|\tilde u^*\Xi - \Xi\|_{L^3(Y)} \;=\; \|\mathrm{d}\tilde w\|_{L^3(Y_{\bar\tau})} \;\le\; C\|u^*\Xi|_\Sigma - \Xi|_\Sigma\|_{L^2(\Sigma)}.$$

Finally, we need a smooth approximation of $\tilde u$ that so far is only continuous. In case $\|u^*\Xi|_\Sigma - \Xi|_\Sigma\|_{L^2(\Sigma)} = 0$ we have $\mathrm{d}\tilde u = \tilde u\Xi - \Xi\tilde u$, where Ξ is smooth, so automatically $\tilde u \in \mathcal{G}(Y)$. Otherwise we can find a smooth approximation $\bar u \in \mathcal{G}(Y)$ of the map $\tilde u \in W^{1,\frac32 p}(Y, \mathrm{G}) \subset \mathcal{C}^0(Y, \mathrm{G})$ with fixed boundary values[7]

[7] Pick any extension $v \in \mathcal{C}^\infty(Y, \mathrm{G})$ of $\tilde u|_{\partial Y}$. Then $\tilde u - v \in W^{1,\frac32 p}(Y, \mathbb{R}^k)$ has zero boundary values and thus can be approximated by $w \in \mathcal{C}_0^\infty(Y, \mathbb{R}^k)$. Now $v + w \in \mathcal{C}^\infty(Y, \mathbb{R}^k)$ is $\mathcal{C}^0$-close to $\tilde u$ and already identic to it on ∂Y. So a projection from a neighbourhood of $\mathrm{G} \subset \mathbb{R}^k$ to G composed with $v + w$ yields the required approximation.

$\bar{u}|_{\partial Y} = \tilde{u}|_{\partial Y} = u$ and $\|\bar{u}-\tilde{u}\|_{W^{1,\frac{3}{2}p}(Y,\mathbb{R}^k)} \leq \min(1\,,\,\|u^*\Xi|_\Sigma - \Xi|_\Sigma\|_{L^2(\Sigma)})$. This is an approximation in $W^{1,3}(Y)$ as well as $\mathcal{C}^0(Y)$. So we introduce the notation $\mathrm{d}_\Xi\tilde{u} = \tilde{u}(\tilde{u}^*\Xi - \Xi) = \mathrm{d}\tilde{u} + \Xi\tilde{u} - \tilde{u}\Xi$ to estimate

$$
\begin{aligned}
&\|\bar{u}^*\Xi - \Xi\|_{L^3(Y)} \\
&\leq \|\tilde{u}^*\Xi - \Xi\|_{L^3(Y)} + \big\|\bar{u}^{-1}\big(\mathrm{d}_\Xi\bar{u} - \mathrm{d}_\Xi\tilde{u}\big)\big\|_{L^3(Y)} + \big\|\big(\bar{u}^{-1} - \tilde{u}^{-1}\big)\mathrm{d}_\Xi\tilde{u}\|_{L^3(Y)} \\
&\leq C\|u^*\Xi|_\Sigma - \Xi|_\Sigma\|_{L^2(\Sigma)}.
\end{aligned}
$$

The constant C again depends on $\Xi \in \mathcal{A}^{0,3}(Y)$, but since this only varies in a compact set, it can be chosen uniform.

$\square$

Proof of lemma 1.6 (i):
For a given smooth path $A : (-\varepsilon,\varepsilon) \to \mathcal{L}_Y \cap \mathcal{A}(\Sigma)$ let $\Theta = (\theta_i) \in \mathcal{C}^\infty((-\varepsilon,\varepsilon), \mathrm{G}^g)$ be given by $\theta_i(s) = \mathrm{hol}_{\tilde{\alpha}_i(\tau)}(A(s))$. We pick $\tau \in [-1,1]$ as in lemma 3.1 (i), so

$$
\big|\partial_s\Theta(0)\big|_{\mathrm{T}_{\Theta(0)}\mathrm{G}^g} \leq C\|\partial_s A(0)\|_{L^1(\Sigma)}.
$$

We can also pick one of the fixed $\Theta_j^0 \in \mathrm{G}^g$ with $\Theta(s) \in B_\Delta(\Theta_j^0)$ for all $s \in (-\varepsilon,\varepsilon)$ for some smaller $\varepsilon > 0$. (Note that it suffices to construct $\tilde{A}(s) \in \mathcal{A}_{\mathrm{flat}}(Y)$ for a neighbourhood of $s = 0$. Then we can arbitrarily extend it to a larger interval.) Now we can use the chart (12) with $z = z(\tau)$ to write $A(s) = u(s)^*\Xi_j(\Theta(s))|_\Sigma$ with a smooth path $u : (-\varepsilon,\varepsilon) \to \mathcal{G}_z(\Sigma)$. So we have

$$
\partial_s A(0) = u(0)^{-1}\big(\mathrm{T}_{\Xi_0}\Xi_j(\partial_s\Theta(0))\big|_\Sigma + \mathrm{d}^\Sigma_{\Xi_0}\xi\big)\,u(0)
$$

with $\xi = \partial_s u(0)u(0)^{-1} \in \mathcal{C}^\infty_z(\Sigma,\mathfrak{g})$ and $\Xi_0 = \Xi_j(\Theta(0))$. Here the operators $\mathrm{T}_{\Xi_0}\Xi_j|_\Sigma : \mathrm{T}B_\Delta(\Theta_j^0) \to L^2(\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g})$ bounded, and we can choose a uniform constant on all of $B_\Delta(\Theta_j^0)$ for all $j = 1,\dots,N$. So we have with another uniform constant C

$$
\big\|\mathrm{d}^\Sigma_{\Xi_0}\xi\big\|_{L^2\Sigma} \;\leq\; \big\|\partial_s A(0)\big\|_{L^2\Sigma} + \big\|\mathrm{T}_{\Xi_0}\Xi_j|_\Sigma\big\|\big|\partial_s\Theta(0)\big|_{\mathrm{T}_{\Theta(0)}\mathrm{G}^g} \;\leq\; C\big\|\partial_s A(0)\big\|_{L^2\Sigma}.
$$

Next, lemma 3.2 provides $\tilde{u}_0 \in \mathcal{G}(Y)$ with $\tilde{u}_0|_{\partial Y} = u$ and $\tilde{\xi} \in \mathcal{C}^\infty(Y,\mathfrak{g})$ such that $\tilde{\xi}|_{\partial Y} = \xi$ and

$$
\|\mathrm{d}_{\Xi_0}\tilde{\xi}\|_{L^3(Y)} \;\leq\; C\|\mathrm{d}^\Sigma_{\Xi_0}\xi\|_{L^2(\Sigma)}.
$$

If we now define $\tilde{A} : (-\varepsilon,\varepsilon) \to \mathcal{A}_{\mathrm{flat}}(Y)$ by $\tilde{A}(s) = (\exp(s\tilde{\xi})\,\tilde{u}_0)^*\Xi_j(\Theta(s))$, then indeed $\partial_s\tilde{A}(0)|_{\partial Y} = \partial_s A(0)$ and

$$
\begin{aligned}
\|\partial_s\tilde{A}(0)\|_{L^3(Y)} \;&=\; \big\|\tilde{u}_0^{-1}\big(\mathrm{T}_{\Xi_0}\Xi_j(\partial_s\Theta(0)) + \mathrm{d}_{\Xi_0}\tilde{\xi}\big)\,\tilde{u}_0\big\|_{L^3(Y)} \\
&\leq\; \big\|\mathrm{T}_{\Xi_0}\Xi_j\big\||\partial_s\Theta(0)| + \|\mathrm{d}_{\Xi_0}\tilde{\xi}\|_{L^3(Y)} \;\leq\; C\|\partial_s A(0)\|_{L^2(\Sigma)}.
\end{aligned}
$$

Here we have moreover chosen a uniform constant of continuity for the operators $\mathrm{T}_{\Xi_0}\Xi_j : \mathrm{T}B_\Delta(\Theta_j^0) \to L^3(Y, \mathrm{T}^*Y\otimes\mathfrak{g})$ on the compact domains $B_\Delta(\Theta_j^0)$ for all $j = 1,\dots,N$. $\square$

Proof of lemma 1.6 (ii):
Let $A_0, A_1 \in \mathcal{L}_Y \cap \mathcal{A}(\Sigma)$ be given. We will prove the lemma by construction, assuming that $A_0 = \Xi_j(\Phi^0)|_\Sigma$ for some $\Phi^0 = (\phi_i^0) \in B_{\frac{1}{2}\Delta}(\Theta_j^0)$.

In general, we have $u_0 \in \mathcal{G}(\Sigma)$ such that $A_0 = u_0^*\Xi_j(\Phi^0)|_\Sigma$. The construction below then gives extensions $\tilde{A}_0, \tilde{A}_1 \in \mathcal{A}_{\rm flat}(Y)$ of $(u_0^{-1})^*A_0$ and $(u_0^{-1})^*A_1$. Moreover, lemma 3.1 provides $\tilde{u}_0 \in \mathcal{G}(Y)$ such that $\tilde{u}_0|_{\partial Y} = u_0$. Then $\tilde{u}_0^*\tilde{A}_0$ and $\tilde{u}_0^*\tilde{A}_1$ are extensions of A_0 and A_1, and the estimate on $\tilde{A}_0 - \tilde{A}_1$ also yields

$$
\begin{aligned}
\big\|\tilde{u}_0^*\tilde{A}_0 - \tilde{u}_0^*\tilde{A}_1\big\|_{L^3(Y)} &= \|\tilde{A}_0 - \tilde{A}_1\|_{L^3(Y)} \\
&\leq C_Y \big\|(u_0^{-1})^*A_0 - (u_0^{-1})^*A_1\big\|_{L^2(\Sigma)} = C_Y\|A_0 - A_1\|_{L^2(\Sigma)}.
\end{aligned}
$$

So from now on suppose that $A_0 = \Xi_j(\Phi^0)|_\Sigma$. Then we already have the extension $\tilde{A}_0 := \Xi_j(\Phi^0) \in \mathcal{A}_{\rm flat}(Y)$. Note that $\mathrm{hol}_{\tilde{\alpha}_i(\tau)}(A_0) = \phi_i^0$ for all $\tau \in [-1,1]$. Lemma 3.1 (ii) then provides $\tau \in [-1,1]$ such that for all $i = 1, \ldots, g$

$$
\mathrm{dist}_{\mathrm{G}}\big(\phi_i^0\,,\, \mathrm{hol}_{\tilde{\alpha}_i(\tau)}(A_1)\big) \;\leq\; C\|A_0 - A_1\|_{L^1(\Sigma)}.
$$

If $\|A_0 - A_1\|_{L^1(\Sigma)} \leq \frac{\Delta}{2C}$ then this implies $\Phi := (\mathrm{hol}_{\tilde{\alpha}_i(\tau)}(A_1))_{i=1,\ldots,g} \in B_\Delta(\Theta_j^0)$. In that case we have found a flat connection $\tilde{A} := \Xi_j(\Phi)$ on Y whose holonomies (based at $z(\tau)$) coincide with those of A_1, and

$$
\begin{aligned}
&\|\tilde{A}_0 - \tilde{A}\|_{L^3(Y)} + \|A_0 - \tilde{A}|_\Sigma\|_{L^2(\Sigma)} \\
&= \big\|\Xi_j(\Phi_0) - \Xi_j(\Phi)\big\|_{L^3(Y)} + \big\|\big(\Xi_j(\Phi_0) - \Xi_j(\Phi)\big)|_\Sigma\big\|_{L^2(Y)} \\
&\leq C\,\mathrm{dist}_{\mathrm{G}^g}\big(\Phi_0\,,\,\Phi\big) \;\leq\; C\|A_0 - A_1\|_{L^1(\Sigma)}.
\end{aligned} \tag{14}
$$

Here and in the following, all uniform constants are denoted by C. We have in particular used the fact that the sections Ξ_j and $\Xi_j|_\Sigma$ are smooth on a compact set, so they are Lipschitz continuous with uniform constants.

In case $\|A_0 - A_1\|_{L^1(\Sigma)} \geq \frac{\Delta}{2C}$ we also use the sections (11) to find a flat connection $\tilde{A} := \Xi_{j'}(\Phi)$ on Y with the same holonomies (based at $z(\tau)$) as A_1. The sections are uniformly bounded in $L^3(Y)$ and $L^2(\Sigma)$ since they are smooth over a union of compact sets. Hence there is a uniform constant $\bar{C}$ such that $\|\tilde{A}_0 - \tilde{A}\|_{L^3(Y)} + \|A_0 - \tilde{A}|_\Sigma\|_{L^2(\Sigma)} \leq \bar{C}$, and thus (14) again holds with $C = \frac{2C\bar{C}}{\Delta}$.

For the two flat connections A_1 and $\tilde{A}|_\Sigma$ with coinciding holonomies one then finds a gauge transformation $u \in \mathcal{G}(\Sigma)$ such that $u^*\tilde{A}|_\Sigma = A_1$. Now by lemma 3.2 (ii) there exists an extension $\tilde{u} \in \mathcal{G}(Y)$ with $\tilde{u}|_\Sigma = u$ and such that

$$
\begin{aligned}
\big\|\tilde{u}^*\tilde{A} - \tilde{A}\big\|_{L^3(Y)} &\leq\; C\|u^*\tilde{A}|_\Sigma - \tilde{A}|_\Sigma\|_{L^2(\Sigma)} \\
&\leq\; C\big(\|A_1 - A_0\|_{L^2(\Sigma)} + \|A_0 - \tilde{A}|_\Sigma\|_{L^2(\Sigma)}\big) \\
&\leq\; C\|A_1 - A_0\|_{L^2(\Sigma)}.
\end{aligned}
$$

So if we put $\tilde{A}_1 := \tilde{u}^*\tilde{A} \in \mathcal{A}_{\rm flat}(Y)$, then indeed $\tilde{A}_1|_{\partial Y} = A_1$ and

$$
\big\|\tilde{A}_1 - \tilde{A}_0\big\|_{L^3(Y)} \;\leq\; \big\|\tilde{u}^*\tilde{A} - \tilde{A}\big\|_{L^3(Y)} + \big\|\tilde{A} - \tilde{A}_0\big\|_{L^3(Y)} \;\leq\; C\|A_1 - A_0\|_{L^2(\Sigma)}.
$$

$\square$

4 Isoperimetric inequalities

The aim of this section is to firstly introduce the local Chern-Simons functional and prove the isoperimetric inequality, lemma 1.8. Secondly, we will show how the Chern-Simons functional is related to the energy of solutions of the boundary value problem (2). This relation will yield a control of the energy that will be the key to the removal of singularities in the next section.

The usual Chern-Simons functional on a closed 3-manifold M is

$$\mathcal{CS}(\Xi) = \tfrac{1}{2}\int_M \langle\, \Xi \wedge \bigl(F_\Xi - \tfrac{1}{6}[\Xi \wedge \Xi]\bigr)\,\rangle \qquad \forall\, \Xi \in \mathcal{A}(M).$$

It is not gauge invariant, but its change $\mathcal{CS}(\Xi) - \mathcal{CS}(u^*\Xi) = 4\pi^2 \deg(u)$ is determined by the degree of the gauge transformation $u \in \mathcal{G}(M)$. The negative gradient flow lines of $\mathcal{CS}$ are the anti-self-dual connections on $\mathbb{R} \times M$. This can be seen from the fact that the differential $\mathrm{d}_\Xi \mathcal{CS} : \Omega^1(M;\mathfrak{g}) \to \mathbb{R}$ is given by $\alpha \;\mapsto\; \int_M \langle\, \alpha \wedge F_\Xi\,\rangle$.

If M is a manifold with boundary, then this 1-form is not closed – its differential is the natural symplectic structure on $\Omega^1(\partial M;\mathfrak{g})$, c.f. [Sa]. So it is natural to impose Lagrangian boundary conditions $\Xi|_{\partial M} \in \mathcal{L}$. On this subset of connections, the above 1-form is closed. However it is only the differential of a multi-valued functional. If the Lagrangian is $\mathcal{L}_Y$, given by the flat connections on a handle body Y restricted to the boundary $\partial Y = \Sigma$, then this multi-valued Chern-Simons functional can be represented as follows. Given $\Xi \in \mathcal{A}(M)$ with $\Xi|_{\partial M} \in \mathcal{L}_Y$ one can find $\tilde\Xi \in \mathcal{A}_{\mathrm{flat}}(Y)$ with $\tilde\Xi|_{\partial Y} = \Xi|_{\partial M}$ and use this to define

$$\mathcal{CS}_{\mathcal{L}_Y}(\Xi) = \tfrac{1}{2}\int_M \langle\, \Xi \wedge \bigl(F_\Xi - \tfrac{1}{6}[\Xi \wedge \Xi]\bigr)\,\rangle + \tfrac{1}{12}\int_Y \langle\, \tilde\Xi \wedge [\tilde\Xi \wedge \tilde\Xi]\,\rangle.$$

This is the actual Chern-Simons functional on the closed manifold $M \cup_\Sigma \bar{Y}$ (where $\bar{Y}$ has the reversed orientation) of the connection given by Ξ and $\tilde\Xi$ on the two parts. It is welldefined only up to multiples of $4\pi^2$ due to the choice of different extensions $\tilde\Xi$ of $\Xi|_{\partial M}$. A change of this extension corresponds to the action of a gauge transformation on $M \cup_\Sigma \bar{Y}$ that is trivial on M. (The gauge equivalence class of a flat connection on Y is fixed by its holonomies on ∂Y.)

Our energy identities below deal with connections $\Xi \in \mathcal{A}([0,\pi]\times\Sigma)$ with boundary values $\Xi|_{\phi=0}, \Xi|_{\phi=\pi} \in \mathcal{L}_Y$. These can be put into the special gauge $\Xi = A$ with $A : [0,\pi] \to \mathcal{A}(\Sigma)$. So equivalently to $\mathcal{CS}_{\mathcal{L}_Y}(\Xi)$, we can define the **local Chern-Simons functional** for smooth paths $A : [0,\pi] \to \mathcal{A}(\Sigma)$ with endpoints $A(0), A(\pi) \in \mathcal{L}_Y$ (that will actually be welldefined for short paths):

$$\begin{aligned}\mathcal{CS}(A) = &-\tfrac{1}{2}\int_0^\pi \int_\Sigma \langle\, A \wedge \partial_\phi A\,\rangle \,\mathrm{d}\phi \\ &- \tfrac{1}{12}\int_Y \langle\, \tilde A(0) \wedge [\tilde A(0) \wedge \tilde A(0)]\,\rangle + \tfrac{1}{12}\int_Y \langle\, \tilde A(\pi) \wedge [\tilde A(\pi) \wedge \tilde A(\pi)]\,\rangle,\end{aligned} \tag{15}$$

where $\tilde A(0), \tilde A(\pi) \in \mathcal{A}_{\mathrm{flat}}(Y)$ such that $\tilde A(0)|_{\partial Y} = A(0)$, $\tilde A(\pi)|_{\partial Y} = A(\pi)$, and

$$\|\tilde A(0) - \tilde A(\pi)\|_{L^3(Y)} \le C_Y \|A(0) - A(\pi)\|_{L^2(\Sigma)}. \tag{16}$$

Here C_Y is the constant from lemma 1.6 (ii), which ensures the existence of the extensions $\tilde{A}(0)$ and $\tilde{A}(\pi)$. This $\mathcal{CS}(A)$ equals the above $\mathcal{CS}_{\mathcal{L}_Y}(\Xi)$ in the special gauge. So a priori it is defined only up to multiples of $4\pi^2$ due to the freedom in the choice of the extensions $\tilde{A}(0), \tilde{A}(\pi)$. However, we will see below that for sufficiently small $\int_0^\pi \|\partial_\phi A\|_{L^2(\Sigma)}$ this Chern-Simons functional is welldefined, i.e. any choice of extensions $\tilde{A}(\pi), \tilde{A}(0)$ that satisfies (16) will give the same value for $\mathcal{CS}(A)$.

Proof of lemma 1.8 : Let $A : [0,\pi] \to \mathcal{A}(\Sigma)$ be a smooth path with $A(0), A(\pi) \in \mathcal{L}_Y$ and $\int_0^\pi \|\partial_\phi A\|_{L^2(\Sigma)} \leq \varepsilon$, where $\varepsilon > 0$ will be fixed later on. Consider any flat connections $\tilde{A}(0), \tilde{A}(\pi) \in \mathcal{A}_{\text{flat}}(Y)$ such that $\tilde{A}(0)|_{\partial Y} = A(0)$, $\tilde{A}(\pi)|_{\partial Y} = A(\pi)$, and (16) holds. With these we calculate

$$\begin{aligned}
&\int_0^\pi \int_\Sigma \langle\, A \wedge \partial_\phi A \,\rangle \,\mathrm{d}\phi \\
&= \int_0^\pi \int_\Sigma \langle\, \big(A(0) + \textstyle\int_0^\phi \partial_\phi A(\theta)\,\mathrm{d}\theta\big) \wedge \partial_\phi A(\phi) \,\rangle \,\mathrm{d}\phi \\
&= \int_0^\pi \int_0^\phi \int_\Sigma \langle\, \partial_\phi A(\theta) \wedge \partial_\phi A(\phi) \,\rangle \,\mathrm{d}\theta\,\mathrm{d}\phi + \int_\Sigma \langle\, A(0) \wedge A(\pi) \,\rangle - \langle\, A(0) \wedge A(0) \,\rangle \\
&= \int_0^\pi \int_0^\phi \int_\Sigma \langle\, \partial_\phi A(\theta) \wedge \partial_\phi A(\phi) \,\rangle \,\mathrm{d}\theta\,\mathrm{d}\phi + \int_Y \langle\, \mathrm{d}\tilde{A}(0) \wedge \tilde{A}(\pi) \,\rangle - \langle\, \tilde{A}(0) \wedge \mathrm{d}\tilde{A}(\pi) \,\rangle
\end{aligned}$$

Now use the fact that $F_{\tilde{A}(0)} = F_{\tilde{A}(\pi)} = 0$ and choose $\varepsilon \leq \frac{6}{C_Y^3}$ to obtain

$$\begin{aligned}
\mathcal{CS}(A) &= -\tfrac{1}{2} \int_0^\pi \int_0^\phi \int_\Sigma \langle\, \partial_\phi A(\theta) \wedge \partial_\phi A(\phi) \,\rangle \,\mathrm{d}\theta\,\mathrm{d}\phi \\
&\quad + \tfrac{1}{4} \int_Y \langle\, [\tilde{A}(0) \wedge \tilde{A}(0)] \wedge \tilde{A}(\pi) \,\rangle - \langle\, \tilde{A}(0) \wedge [\tilde{A}(\pi) \wedge \tilde{A}(\pi)] \,\rangle \\
&\quad - \tfrac{1}{12} \int_Y \langle\, [\tilde{A}(0) \wedge \tilde{A}(0)] \wedge \tilde{A}(0) \,\rangle - \langle\, [\tilde{A}(\pi) \wedge \tilde{A}(\pi)] \wedge \tilde{A}(\pi) \,\rangle \\
&= -\tfrac{1}{2} \int_0^\pi \int_0^\phi \int_\Sigma \langle\, \partial_\phi A(\theta) \wedge \partial_\phi A(\phi) \,\rangle \,\mathrm{d}\theta\,\mathrm{d}\phi \\
&\quad - \tfrac{1}{12} \int_Y \langle\, \big[(\tilde{A}(0) - \tilde{A}(\pi)) \wedge (\tilde{A}(0) - \tilde{A}(\pi))\big] \wedge \big(\tilde{A}(0) - \tilde{A}(\pi)\big) \,\rangle \\
\Rightarrow \big|\mathcal{CS}(A)\big| &\leq \tfrac{1}{2} \left(\int_0^\pi \big\|\partial_\phi A\big\|_{L^2(\Sigma)} \,\mathrm{d}\phi \right)^2 + \tfrac{1}{12} \left(\big\|\tilde{A}(0) - \tilde{A}(\pi)\big\|_{L^3(Y)} \right)^3 \\
&\leq \left(\tfrac{1}{2} + \tfrac{C_Y^3}{12} \big\|A(0) - A(\pi)\big\|_{L^2(\Sigma)} \right) \left(\int_0^\pi \big\|\partial_\phi A\big\|_{L^2(\Sigma)} \,\mathrm{d}\phi \right)^2 \\
&\leq \left(\int_0^\pi \big\|\partial_\phi A\big\|_{L^2(\Sigma)} \,\mathrm{d}\phi \right)^2 \;\leq\; \varepsilon^2.
\end{aligned}$$

If we choose $\varepsilon > 0$ small enough, then this implies that our choice of extensions will always yield values $\mathcal{CS}(A) \in [-\pi^2, \pi^2]$. As seen before, $\mathcal{CS}(A)$ is the usual

Chern-Simons functional on the closed 3-manifold $\bar{Y} \cup_{\{\pi\}\times\Sigma} [0,\pi]\times\Sigma \cup_{\{0\}\times\Sigma} Y$ of the connection given by $\tilde{A}(\pi)$, A, and $\tilde{A}(0)$ on the different parts. If we change the extensions $\tilde{A}(0)$ and $\tilde{A}(\pi)$, then this corresponds to changing the connection on the closed manifold by one gauge transformation (that is nontrivial only in the interior of Y and $\bar{Y}$). Hence the Chern-Simons functional will change by a multiple (the degree of the gauge transformation) of $4\pi^2$. This cannot lead to another value in the interval $[-\pi^2, \pi^2]$, hence the value of $\mathcal{CS}(A)$ is uniquely determined by the condition (16) on the extensions. □

The Chern-Simons functional is the starting point for the removal of singularities in theorem 1.5 and remark 1.9. In both cases, the energy on a neighbourhood of the singularity can be expressed by the Chern-Simons functional (of the connection on the boundary of this neighbourhood in a certain gauge). This will yield a control on the energy near the singularity. In the interior case, remark 1.9, we fix the radius $r_0 > 0$ and a metric of normal type on $B \times \Sigma$. We use the following notation for circles and punctured balls centered at 0,

$$S_r := \partial B_r, \qquad B_r^* := B_r(0) \setminus \{0\} \subset \mathbb{R}^2, \qquad B^* := B_{r_0}^*.$$

We then consider a connection $\Xi \in \mathcal{A}(B^* \times \Sigma)$ that is anti-self-dual,

$$*F_\Xi + F_\Xi = 0. \tag{17}$$

Using polar coordinates $r \in (0, r_0]$, $\phi \in [0, 2\pi]$ on B^* we assume as in remark 1.9 that the connection is in the gauge $\Xi = A + R\mathrm{d}r$ with no $\mathrm{d}\phi$-component and $A : D \to \Omega^1(\Sigma, \mathfrak{g})$, $R : D \to \Omega^0(\Sigma, \mathfrak{g})$. Then (17) then identifies the curvature components

$$*F_A \;=\; r^{-1}\partial_\phi R\,, \qquad *\bigl(\partial_r A - \mathrm{d}_A R\bigr) \;=\; r^{-1}\partial_\phi A.$$

Hence for $0 < \rho \leq r_0$ the energy of the connection on $B_\rho^* \times \Sigma$ is

$$\mathcal{E}(\rho) := \tfrac{1}{2}\int_{B_\rho^*\times\Sigma} |F_\Xi|^2 = \int_0^\rho \int_0^{2\pi} \bigl(\|F_A\|_{L^2(\Sigma)}^2 + r^{-2}\|\partial_\phi A\|_{L^2(\Sigma)}^2\bigr)\, r\,\mathrm{d}\phi\,\mathrm{d}r. \tag{18}$$

We shall see in lemma 4.1 (i) that in this gauge the Chern-Simons functional on $S_r \times \Sigma$ equals the energy $\mathcal{E}(r)$, which leads to a decay estimate for the energy.

In the boundary case, theorem 1.5, we fix a radius $r_0 > 0$ and a metric of normal type on $D \times \Sigma$, and we denote the punctured half balls by

$$D_r^* := B_r(0) \setminus \{0\} \cap \mathbb{H}^2, \qquad D^* := D_{r_0}^*.$$

We consider a connection $\Xi \in \mathcal{A}(D^* \times \Sigma)$ that solves the boundary value problem

$$\left\{ \begin{array}{l} *F_\Xi + F_\Xi = 0, \\ \Xi|_{(s,0)\times\Sigma} \in \mathcal{L}_Y \quad \forall s \in [-r_0, 0) \cup (0, r_0]. \end{array} \right. \tag{19}$$

Using polar coordinates $r \in (0, r_0]$, $\phi \in [0, \pi]$ on D^* we can always choose a gauge $\Xi = A + R\mathrm{d}r$ with no $\mathrm{d}\phi$-component. Then the energy function is

$$\mathcal{E}(\rho) := \tfrac{1}{2}\int_{D_\rho^*\times\Sigma} |F_\Xi|^2 \;\; = \int_0^\rho \int_0^{\pi} \bigl(\|F_A\|_{L^2(\Sigma)}^2 + r^{-2}\|\partial_\phi A\|_{L^2(\Sigma)}^2\bigr)\, r\,\mathrm{d}\phi\,\mathrm{d}r. \tag{20}$$

We shall see that for sufficiently small $\rho > 0$ this energy equals the local Chern-Simons functional $\mathcal{CS}(A(\rho,\cdot))$, and this yields a decay estimate for the energy.

Lemma 4.1

(i) Let $\Xi \in \mathcal{A}(B^* \times \Sigma)$ satisfy (17) and $\mathcal{E}(r_0) < \infty$, and suppose that it is in the gauge $\Xi = A + R\mathrm{d}r$ with $\Phi \equiv 0$. Then for all $r \le r_0$

$$\mathcal{E}(r) \;=\; -\mathcal{CS}(\Xi|_{S_r\times\Sigma}) \;\le\; \tfrac{1}{2}\left(\int_0^{2\pi} \left\|\partial_\phi A(r,\phi)\right\|_{L^2(\Sigma)} \mathrm{d}\phi\right)^2 \;\le\; \pi r\,\dot{\mathcal{E}}(r)$$

and hence $\mathcal{E}(r) \le C r^{2\beta}$ with $\beta = \frac{1}{2\pi} > 0$ and some constant C.

(ii) Let $\Xi \in \mathcal{A}(D^* \times \Sigma)$ satisfy (19) and $\mathcal{E}(r_0) < \infty$, and suppose that it is in the gauge $\Xi = A + R\mathrm{d}r$ with $\Phi \equiv 0$. Then there exists $0 < r_1 \le r_0$ such that for all $r \le r_1$

$$\mathcal{E}(r) \;=\; -\mathcal{CS}(A(r,\cdot)) \;\le\; \left(\int_0^{\pi} \left\|\partial_\phi A(r,\phi)\right\|_{L^2(\Sigma)} \mathrm{d}\phi\right)^2 \;\le\; \pi r\,\dot{\mathcal{E}}(r)$$

and hence $\mathcal{E}(r) \le C r^{2\beta}$ with $\beta = \frac{1}{2\pi} > 0$ and some constant C.

Note that for every connection on $B^*\times\Sigma$ (and similarly for $D^*\times\Sigma$) with finite energy the decay of the energy $\mathcal{E}(r) \to 0$ as $r \to 0$ is automatic: The assumption $\mathcal{E}(r_0) < \infty$ just means that the limit $\frac{1}{2}\int_{(B_{r_0}\setminus B_r)\times\Sigma} |F_\Xi|^2 = \mathcal{E}(r_0) - \mathcal{E}(r) \to \mathcal{E}(r_0)$ exists as $r \to 0$. Now this lemma allows to control the rate of decay of $\mathcal{E}(r)$ for anti-self-dual connections or solutions of the boundary value problem (19).

The proof of lemma 4.1 will make use of lemma 5.4, which implies that

$$\int r^2 \|F_\Xi(r,\phi)\|^2_{L^2(\Sigma)}\,\mathrm{d}\phi \;\le\; C\,\mathcal{E}(2r) \;\underset{r\to 0}{\longrightarrow}\; 0.$$

For any smooth connection with finite energy there always exists a sequence $r_i \to 0$ for which the above integral converges to zero. This suffices for the proof of lemma 4.1 (i), but in case (ii) we need this control for all sufficiently small $r > 0$ in order to be able to use the local Chern-Simons functional. Lemma 4.1 will only be used for the proof of theorem 1.5 and remark 1.9 and does not affect the other results in section 5, so we can indeed use lemma 5.4 in its proof.

Proof of lemma 4.1: We start with the interior case (i). Let $0 < \rho \le r_0$, then by assumption $\mathcal{E}(\rho) \le \mathcal{E}(r_0)$ is finite, i.e. it exists as the limit

$$\mathcal{E}(\rho) \;=\; \lim_{\delta\to 0} \tfrac{1}{2}\int_{(B_\rho\setminus B_\delta)\times\Sigma} |F_\Xi|^2.$$

Due to the anti-self-duality of F_Ξ we can rewrite

$$\begin{aligned}
\tfrac{1}{2}\int_{(B_\rho\setminus B_\delta)\times\Sigma} |F_\Xi|^2 &= -\tfrac{1}{2}\int_{(B_\rho\setminus B_\delta)\times\Sigma} \langle\, F_\Xi \wedge F_\Xi \,\rangle \\
&= -\tfrac{1}{2}\int_{(B_\rho\setminus B_\delta)\times\Sigma} \mathrm{d}\,\langle\, \Xi \wedge \left(F_\Xi - \tfrac{1}{6}[\Xi\wedge\Xi]\right) \rangle \\
&= -\mathcal{CS}(\Xi|_{S_\rho\times\Sigma}) + \mathcal{CS}(\Xi|_{S_\delta\times\Sigma}).
\end{aligned}$$

Here the Chern-Simons functional on $S_r\times\Sigma$ for $r=\rho$ and $r=\delta$ is not gauge invariant but changes by multiples of $4\pi^2$ under gauge transformations of nonzero degree. However, the special gauge $\Xi|_{S_r\times\Sigma}=A(r,\cdot):[0,2\pi]\to\mathcal{A}(\Sigma)$ fixes these values, and we obtain

$$\begin{aligned}\mathcal{CS}(\Xi|_{S_r\times\Sigma}) &= -\tfrac12\int_0^{2\pi}\int_\Sigma\langle\, A\wedge\partial_\phi A\,\rangle\,\mathrm{d}\phi\\ &= -\tfrac12\int_0^{2\pi}\int_\Sigma\langle\,\Bigl(A(r,0)+\int_0^\phi\partial_\phi A(r,\theta)\,\mathrm{d}\theta\Bigr)\wedge\partial_\phi A(r,\phi)\,\rangle\,\mathrm{d}\phi\\ &= -\tfrac12\int_0^{2\pi}\int_0^\phi\int_\Sigma\langle\,\partial_\phi A(r,\theta)\wedge\partial_\phi A(r,\phi)\,\rangle\,\mathrm{d}\theta\,\mathrm{d}\phi.\end{aligned}$$

Hence for all $0<r\le r_0$

$$\begin{aligned}2\,\bigl|\mathcal{CS}(\Xi|_{S_r\times\Sigma})\bigr| &\le \Bigl(\int_0^{2\pi}\bigl\|\partial_\phi A(r,\phi)\bigr\|_{L^2(\Sigma)}\,\mathrm{d}\phi\Bigr)^2\\ &\le \pi\int_0^{2\pi}\bigl\|\partial_\phi A(\rho,\phi)\bigr\|^2_{L^2(\Sigma)}\,\mathrm{d}\phi \;\le\; \tfrac12\pi\int_0^{2\pi}\rho^2\bigl\|F_\Xi(\rho,\phi)\bigr\|^2_{L^2(\Sigma)}\,\mathrm{d}\phi.\end{aligned}$$

Now we know by lemma 5.4 that the last expression (and thus also the length of the path $A(r,\cdot)\in\mathcal{A}^{0,2}(\Sigma)$) goes to zero as $r\to 0$. Thus we obtain

$$\mathcal{E}(\rho) \;=\; -\mathcal{CS}(\Xi|_{S_\rho\times\Sigma}) \;\le\; \tfrac12\pi\int_0^{2\pi}\rho^2\bigl\|F_\Xi(\rho,\phi)\bigr\|^2_{L^2(\Sigma)}\,\mathrm{d}\phi \;=\; \pi\rho\,\dot{\mathcal{E}}(\rho),$$

$$\Rightarrow\;\ln\mathcal{E}(r)\;\le\;\ln\mathcal{E}(r_0)-\int_r^{r_0}(\pi\rho)^{-1}\mathrm{d}\rho\;=\;\ln\mathcal{E}(r_0)-\tfrac1\pi\ln r_0+\tfrac1\pi\ln r. \qquad (21)$$

Hence we have $\mathcal{E}(r)\le Cr^{2\beta}$ with $\beta=\frac{1}{2\pi}>0$, which proves (i).

In (ii) we also have for all $0<\rho\le r_1$ (where $r_1>0$ will be fixed later on)

$$\mathcal{E}(\rho)\;=\;\lim_{\delta\to0}\tfrac12\int_{(D_\rho\setminus D_\delta)\times\Sigma}|F_\Xi|^2.$$

We aim to express this as the difference of a functional at $r=\rho$ and at $r=\delta$. The straightforward approach as in (i) would pick up additional boundary terms on $\{\phi=0\}$ and $\{\phi=\pi\}$. We eliminate these by glueing Y to $\Sigma=\partial Y$ and extending the connections $A(r,0),A(r,\pi)\in\mathcal{L}_Y$ to flat connections on Y. More precisely, the oriented boundary of $(D_\rho\setminus D_\delta)\times\Sigma$ consists of $\{r=\rho\}\cong[0,\pi]\times\Sigma$ and $\{r=\delta\}\cong[0,\pi]\times\bar\Sigma$ and the additional parts $\{\phi=0\}\cong[\delta,\rho]\times\Sigma$ and $\{\phi=\pi\}\cong[\delta,\rho]\times\bar\Sigma$ (where $\bar\Sigma$ has the reversed orientation). So we glue in $[\delta,\rho]\times Y$ and $[\delta,\rho]\times\bar Y$ to obtain the smooth 4-manifold

$$X(\delta,\rho)=[\delta,\rho]\times\bar Y\cup_{\{\phi=\pi\}}(D_\rho\setminus D_\delta)\times\Sigma\cup_{\{\phi=0\}}[\delta,\rho]\times Y$$

which has the boundary component $\bar Y\cup_{\{\phi=\pi\}}[0,\pi]\times\Sigma\cup_{\{\phi=0\}}\times Y$ at $r=\rho$ and with reversed orientation at $r=\delta$.

Next, $A(\cdot,0)$ and $A(\cdot,\pi)$ are smooth paths in $\mathcal{L}_Y \cap \mathcal{A}(\Sigma)$. So we can pick smooth paths of extensions $\tilde{A}(\cdot,0), \tilde{A}(\cdot,\pi) : [\delta,\rho] \to \mathcal{A}_{\text{flat}}(Y)$. We also extend the functions $R|_{\phi=0}$ and $R|_{\phi=\pi}$ from $[\delta,\rho]\times\Sigma$ to smooth functions $\tilde{R}_0$ and $\tilde{R}_\pi$ on $[\delta,\rho]\times Y$. These extensions match up to a $W^{1,\infty}$-connection on $X(\delta,\rho)$,

$$\tilde{\Xi} = \begin{cases} \tilde{A}(\cdot,\pi) + \tilde{R}_\pi \mathrm{d}r & ;\text{on } [\delta,\rho]\times\bar{Y}, \\ A + R\mathrm{d}r & ;\text{on } (D_\rho \setminus D_\delta)\times\Sigma, \\ \tilde{A}(\cdot,0) + \tilde{R}_0 \mathrm{d}r & ;\text{on } [\delta,\rho]\times Y. \end{cases}$$

We will choose the two paths of extensions $\tilde{A}(\cdot,0)$ and $\tilde{A}(\cdot,\pi)$ such that for all $\delta \le r \le \rho$ the functional $\mathcal{C}(A(r,\cdot),\tilde{A}(r,0),\tilde{A}(r,\pi))$ given by (15) with these extensions equals the local Chern-Simons functional $\mathcal{CS}(A(r,\cdot))$. For this purpose let $\bar{\varepsilon} > 0$ be the constant from lemma 5.4 and choose $0 < r_1 \le \frac{1}{2}r_0$ such that $\mathcal{E}(2r_1) \le \bar{\varepsilon}$. Then for all $0 < r \le r_1$

$$\begin{aligned}\left(\int_0^\pi \big\|\partial_\phi A(r,\phi)\big\|_{L^2(\Sigma)}\,\mathrm{d}\phi\right)^2 &\le \pi \int_0^\pi \big\|\partial_\phi A(r,\phi)\big\|^2_{L^2(\Sigma)}\,\mathrm{d}\phi \\ &\le \tfrac{\pi}{2}\int_0^\pi r^2 \big\|F_\Xi(r,\phi)\big\|^2_{L^2(\Sigma)}\,\mathrm{d}\phi \;\le\; C\mathcal{E}(2r).\end{aligned}$$

Now choose $r_1 > 0$ even smaller such that $C\mathcal{E}(2r_1) \le \min(\pi^2,\varepsilon^2)$ with $\varepsilon > 0$ from lemma 1.8. Then the lemma applies to $A(r,\cdot)$ for all $0 < r \le r_1$. In particular, since $\rho \le r_1$, we can choose the two paths of extensions to end at $\tilde{A}(\rho,0)$ and $\tilde{A}(\rho,\pi)$, and hence $\mathcal{C}(A(\rho,\cdot),\tilde{A}(\rho,0),\tilde{A}(\rho,\pi)) = \mathcal{CS}(A(\rho,\cdot))$.

Moreover we know that for all $r\in[\delta,\rho]$ the path $A(r,\cdot)$ is sufficiently small for the local Chern-Simons functional $\mathcal{CS}(A(r,\cdot))$ to be defined and take values in $[-\pi^2,\pi^2]$. Now $\mathcal{C}(A(r,\cdot),\tilde{A}(r,0),\tilde{A}(r,\pi))$ is a smooth function of $r\in[\delta,\rho]$ whose values might differ from $\mathcal{CS}(A(r,\cdot))$ by multiples of $4\pi^2$. We have equality at $r=\rho$ and hence by continuity for all $r\in[\delta,\rho]$ as claimed. Thus we actually obtain the local Chern-Simons functional from $\mathcal{CS}(\tilde{\Xi})$ on $\partial X(\delta,\rho)$,

$$\begin{aligned}\tfrac{1}{2}\int_{(D_\rho\setminus D_\delta)\times\Sigma} |F_\Xi|^2 &= -\tfrac{1}{2}\int_{X(\delta,\rho)} \langle\, F_{\tilde{\Xi}} \wedge F_{\tilde{\Xi}}\,\rangle \\ &= -\tfrac{1}{2}\int_{\partial X(\delta,\rho)} \langle\, \tilde{\Xi}\wedge\big(F_{\tilde{\Xi}} - \tfrac{1}{6}[\tilde{\Xi}\wedge\tilde{\Xi}]\big)\,\rangle \\ &= -\mathcal{CS}(A(\rho,\cdot)) + \mathcal{CS}(A(\delta,\cdot)), \end{aligned} \tag{22}$$

Here we have $F_{\tilde{\Xi}}\wedge F_{\tilde{\Xi}} = -|F_\Xi|^2\,\mathrm{dvol}$ on $(D_\rho\setminus D_\delta)\times\Sigma$ and $F_{\tilde{\Xi}}\wedge F_{\tilde{\Xi}} = 0$ on $[\delta,\rho]\times Y$ since $F_{\tilde{\Xi}}$ vanishes on the 3-dimensional slices $\{r\}\times Y$. Now by lemma 1.8

$$\big|\mathcal{CS}(A(r,\cdot))\big| \;\le\; \left(\int_0^\pi \big\|\partial_\phi A(r,\phi)\big\|_{L^2(\Sigma)}\,\mathrm{d}\phi\right)^2 \;\le\; \frac{\pi}{2}\int_0^\pi r^2\big\|F_\Xi(r,\phi)\big\|^2_{L^2(\Sigma)}\,\mathrm{d}\phi.$$

As $r\to 0$ this expression converges to zero by lemma 5.4. Thus for all $0<\rho\le r_1$

$$\mathcal{E}(\rho) \;=\; -\mathcal{CS}(A(\rho,\cdot)) \;\le\; \frac{\pi}{2}\int_0^\pi \rho^2\big\|F_\Xi(\rho,\phi)\big\|^2_{L^2(\Sigma)}\,\mathrm{d}\phi \;\;=\;\; \pi\rho\,\dot{\mathcal{E}}(\rho).$$

As in (21) this implies $\mathcal{E}(r) \le Cr^{2\beta}$ for all $0 < r \le r_1$ with $\beta = \frac{1}{2\pi} > 0$. □

5 Removal of singularities

This section gives the proofs of theorem 1.5 and remark 1.9. We will also prove a more general removable singularity result, theorem 5.3, that does not require the connections to solve an equation but only assumes a decay condition on the curvature. For solutions of (2), as a consequence of the isoperimetric and by the lemma below, this decay condition is equivalent to the connection having finite energy. In the case of interior singularities of anti-self-dual connections the same is true if we assume the existence of a special gauge as in remark 1.9. Throughout this section we fix metrics of normal type on $D \times \Sigma$ and $B \times \Sigma$.

Lemma 5.1 *Let Ξ be a smooth connection on $D^* \times \Sigma$ or $B^* \times \Sigma$. Suppose that it satisfies (19) or (17) respectively. Then the following are equivalent:*

(i) $\mathcal{E}(r) \leq Cr^{2\beta}$ *for all $r \leq r_0$ and some constants C and $\beta > 0$.*

(ii) $\sup_\phi \|F_\Xi(r,\phi)\|_{L^2(\Sigma)} \leq Cr^{\beta-1}$ *for all $r \leq r_0$ and constants C and $\beta > 0$.*

(iii) $\|F_\Xi\|_{L^p} < \infty$ *for some $p > 2$.*

More precisely, (i) and (ii) are equivalent for fixed $\beta > 0$, (i) implies (iii) for $2 < p < \frac{5}{2}$ with $\frac{1}{p} > \frac{2-\beta}{4}$, and (iii) implies (i) with $\beta = 1 - \frac{2}{p}$.

Moreover, (i) implies for some constant C' on $D^* \times \Sigma$ and $B^* \times \Sigma$ respectively

(iv) $\|F_\Xi(r,\phi)\|_{L^\infty(\Sigma)} \leq C' r^{\beta-2}(\sin\phi)^{-2}$ *for all $r \leq r_0$, $\phi \in (0,\pi)$.*

(iv') $\|F_\Xi(r,\phi)\|_{L^\infty(\Sigma)} \leq C' r^{\beta-2}$ *for all $r \leq r_0$, $\phi \in [0,2\pi]$.*

Remark 5.2 *If (19) or (17) in the above lemma are not satisfied, then still $(ii) \Rightarrow (i)$, $(iii) \Rightarrow (i)$, and $(ii)\&(iv) \Rightarrow (iii)$ or $(ii)\&(iv') \Rightarrow (iii)$ respectively.*

We will first show how this lemma and the subsequent theorem imply our main results, and then give all proofs. The following removal of singularities assumes a control of the curvature as given by lemma 4.1 and 5.1 for finite energy solutions of (19) or (17).

Theorem 5.3

(i) Let $\Xi \in \mathcal{A}(B^* \times \Sigma)$ satisfy (ii) and (iv') of lemma 5.1 with some constant $\beta > 0$. Assume in addition that there exists a gauge in which $\Xi = A + R\mathrm{d}r$ with $\Phi \equiv 0$. Let $2 < p < \frac{5}{2}$ such that $\frac{1}{p} > \frac{2-\beta}{4}$. Then there exists $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(B^* \times \Sigma)$ such that $u^*\Xi$ extends to a connection $\tilde{\Xi} \in \mathcal{A}^{1,p}(B \times \Sigma)$.

Moreover, if Ξ is anti-self-dual, then $\tilde{\Xi}$ will also be anti-self-dual.

(ii) Let $\Xi \in \mathcal{A}(D^* \times \Sigma)$ satisfy (ii) and (iv) of lemma 5.1 with some constant $\beta > 0$. Let $2 < p < \frac{5}{2}$ such that $\frac{1}{p} > \frac{2-\beta}{4}$. Then there is $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(D^* \times \Sigma)$ such that $u^*\Xi$ extends to a connection $\tilde{\Xi} \in \mathcal{A}^{1,p}(D \times \Sigma)$.

Moreover, if Ξ satisfies (19), then $\tilde{\Xi}$ will be a solution of (2).

Proof of theorem 1.5 and remark 1.9:
Let $\Xi \in \mathcal{A}(D^* \times \Sigma)$ satisfy (19) and have finite energy $\mathcal{E}(r_0) < \infty$. Then lemma 4.1 (ii) implies that $\mathcal{E}(r) \leq Cr^{2\beta}$ with $\beta > 0$, and hence we also have (ii) and (iv) as in lemma 5.1. Now pick any $2 < p < \frac{5}{2}$, and in case $0 < \beta < 2$ choose it such that $p < \frac{4}{2-\beta}$. Then theorem 5.3 (ii) provides a gauge transformation $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(D^* \times \Sigma)$ such that $u^*\Xi = \tilde{\Xi}|_{D^* \times \Sigma}$, where $\tilde{\Xi} \in \mathcal{A}^{1,p}(D \times \Sigma)$ is a solution of (2). By the regularity [W3, Thm A] for solutions of (2) we can multiply u by another gauge transformation in $\mathcal{G}^{2,p}(D \times \Sigma)$ (hence still $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(D^* \times \Sigma)$) such that $\tilde{\Xi} \in \mathcal{A}(D \times \Sigma)$ is smooth.

Since on $D^* \times \Sigma$ both Ξ and $\tilde{\Xi}$ are smooth and $u^*\Xi = \tilde{\Xi}$ (i.e. $\mathrm{d}u = u\tilde{\Xi} - \Xi u$) we also know that $u \in \mathcal{G}(D^* \times \Sigma)$ is smooth.

The proof of remark 1.9 is exactly the same. Here lemma 4.1 (i) and theorem 5.3 (i) require the assumption that $\Xi \in \mathcal{A}(B^* \times \Sigma)$ is gauge equivalent to a connection with $\Phi \equiv 0$. Moreover, this argument only uses the wellknown regularity theorem for anti-self-dual connections (see e.g. [W1, Thm 9.4]). $\Box$

Lemma 5.1 will be a consequence of the following mean value inequalities.

Lemma 5.4 *There exist constants C and $\varepsilon > 0$ such that the following holds. Let Ξ be a smooth connection on $D^* \times \Sigma$ or $B^* \times \Sigma$ that satisfies (19) or (17) respectively. Suppose that $\mathcal{E}(2r) \leq \varepsilon$ for some $0 < r \leq \frac{1}{2}r_0$, then*

(i) On $D^* \times \Sigma$ and $B^* \times \Sigma$ $\qquad \sup_\phi \|F_\Xi(r,\phi)\|^2_{L^2(\Sigma)} \leq Cr^{-2}\mathcal{E}(2r).$

(ii) On $D^* \times \Sigma$ for all $\phi \in (0,\pi)$ $\qquad \|F_\Xi(r,\phi)\|^2_{L^\infty(\Sigma)} \leq C(r\sin\phi)^{-4}\mathcal{E}(2r).$

(ii') On $B^* \times \Sigma$ for all $\phi \in [0,2\pi]$ $\qquad \|F_\Xi(r,\phi)\|^2_{L^\infty(\Sigma)} \leq Cr^{-4}\mathcal{E}(2r).$

Proof: We prove (i) in three steps and deduce (ii) and (ii') in the fourth.

Step 1: *We find constants C and $\varepsilon > 0$ such that under the above assumptions*

$$\sup_\phi r\,\|F_\Xi(r,\phi)\|_{L^2(\Sigma)} \leq C.$$

Assume that for some fixed $\varepsilon > 0$ (that we shall fix later on) there is no such bound C. Then we find a sequence of smooth connections Ξ^ν on $D^*\times\Sigma$ or $B^*\times\Sigma$ satisfying (19) or (17) respectively, and we find $\bar{r}^\nu \to r^\infty \in [0, \frac{1}{2}r_0]$ and $\bar{\phi}^\nu \to \phi^\infty$ such that $\mathcal{E}^\nu(2\bar{r}^\nu) \leq \varepsilon$ but $\bar{r}^\nu\,\|F_{\Xi^\nu}(\bar{r}^\nu,\bar{\phi}^\nu)\|_{L^2(\Sigma)} \to \infty$. Here $\mathcal{E}^\nu(\cdot)$ denotes the energy function (20) or (18) of Ξ^ν. Given this we can choose $0 < \bar{\varepsilon}^\nu \leq \frac{1}{2}\bar{r}^\nu$ such that $\bar{\varepsilon}^\nu \to 0$ but still $\bar{\varepsilon}^\nu\,\|F_{\Xi^\nu}(\bar{r}^\nu,\bar{\phi}^\nu)\|_{L^2(\Sigma)} \to \infty$. The Hofer trick, lemma 2.6 then yields $0 < \varepsilon^\nu \leq \bar{\varepsilon}^\nu$ (in particular $\varepsilon^\nu \to 0$) and $(r^\nu,\phi^\nu) \to (r^\infty,\phi^\infty)$ such that the following holds: Firstly, with $R^\nu := 2\|F_{\Xi^\nu}(r^\nu,\phi^\nu)\|_{L^2(\Sigma)} \to \infty$ we have

$$\varepsilon^\nu R^\nu \;\geq\; 2\bar{\varepsilon}^\nu\,\|F_{\Xi^\nu}(\bar{r}^\nu,\bar{\phi}^\nu)\|_{L^2(\Sigma)} \;\to\; \infty.$$

Secondly,

$$\left\|F_{\Xi^\nu}(r,\phi)\right\|_{L^2(\Sigma)} \;\leq\; 2\left\|F_{\Xi^\nu}(r^\nu,\phi^\nu)\right\|_{L^2(\Sigma)} \;=\; R^\nu \qquad \forall (r,\phi) \in B_{\varepsilon^\nu}(r^\nu,\phi^\nu).$$

Here $B_{\varepsilon^\nu}(r^\nu,\phi^\nu)$ denotes the Euclidean ball, where just the center (r^ν,ϕ^ν) is given in polar coordinates. It is contained in $B^*_{2\bar r^\nu}$ because $|r^\nu - \bar r^\nu| \leq \bar\varepsilon^\nu \leq \frac12 \bar r^\nu$. Moreover, in the boundary case it is understood to be intersected with D, so it is contained in $D^*_{2\bar r^\nu}$. Now proposition 2.7 (ii) (with a fixed metric and any $\Delta > 0$) provides a constant C such that for all sufficiently large $\nu \in \mathbb{N}$

$$\big\|F_{A^\nu}(r,\phi)\big\|_{L^\infty(\Sigma)} \;\leq\; C(R^\nu)^2 \qquad \forall (r,\phi) \in B_{\frac12\varepsilon^\nu}(r^\nu,\phi^\nu).$$

Putting this into the estimate of lemma 2.3 we obtain on $B_{\frac12\varepsilon^\nu}(r^\nu,\phi^\nu)$

$$\Delta\big\|F_{\Xi^\nu}\big\|^2_{L^2(\Sigma)} \;\leq\; C(R^\nu)^2\big\|F_{\Xi^\nu}\big\|^2_{L^2(\Sigma)}$$

with another constant C, and in the boundary case moreover

$$-\tfrac{\partial}{\partial t}\big|_{t=0}\big\|F_{\Xi^\nu}\big\|^2_{L^2(\Sigma)} \;\leq\; C\big(\big\|F_{\Xi^\nu}\big\|^2_{L^2(\Sigma)} + \big\|F_{\Xi^\nu}\big\|^3_{L^2(\Sigma)}\big).$$

Now we fix $\varepsilon = \frac13\mu C^{-2}$ with the $\mu > 0$ from proposition 2.5. Then due to $\mathcal{E}^\nu(2\bar r^\nu) \leq \varepsilon$ the mean value inequality applies to the functions $\|F_{\Xi^\nu}\|^2_{L^2(\Sigma)}$ and yields with a new constant C'

$$\big\|F_{\Xi^\nu}(r^\nu,\phi^\nu)\big\|^2_{L^2(\Sigma)} \leq C'\big((R^\nu)^2 + (\varepsilon^\nu)^{-2}\big)\int_{B_{\frac12\varepsilon^\nu}(r^\nu,\phi^\nu)}\big\|F_{\Xi^\nu}\big\|^2_{L^2(\Sigma)}.$$

If we moreover choose $\varepsilon \leq \frac{1}{2C'}$, then this implies $2(R^\nu)^2 \leq (R^\nu)^2 + (\varepsilon^\nu)^{-2}$ and thus $(\varepsilon^\nu R^\nu)^2 \leq 1$ in contradiction to $\varepsilon^\nu R^\nu \to \infty$.

Step 2: *We find constants C and $\varepsilon > 0$ such that under the above assumptions*

$$\sup_\phi r^2\,\|F_A(r,\phi)\|_{L^\infty(\Sigma)} \leq C.$$

Again arguing by contradiction we find a sequence of smooth connections Ξ^ν on $D^*\times\Sigma$ or $B^*\times\Sigma$ satisfying (19) or (17) respectively, moreover $r^\nu \to r^\infty \in [0,\frac12 r_0]$ and $\phi^\nu \to \phi^\infty$ such that $\mathcal{E}^\nu(2r^\nu) \leq \varepsilon$ but $(r^\nu)^2\,\|F_{A^\nu}(r^\nu,\phi^\nu)\|_{L^\infty(\Sigma)} \to \infty$.

Let $0 < \varepsilon^\nu \leq \frac12 r^\nu$, then we know from step 1 that for some $\Delta > 0$

$$\|F_{\Xi^\nu}(r,\phi)\|_{L^2(\Sigma)} \leq 2(r^\nu)^{-1}\Delta \qquad \forall (r,\phi) \in B_{\varepsilon^\nu}(r^\nu,\phi^\nu).$$

Now choose $R^\nu \geq 2(r^\nu)^{-1}\Delta$ such that $R^\nu \to \infty$, then the above is true with $\varepsilon^\nu = \Delta(R^\nu)^{-1} \leq \frac12 r^\nu$. Furthermore, $\varepsilon^\nu \to 0$ and $\varepsilon^\nu R^\nu = \Delta > 0$. So proposition 2.7 (ii) asserts that for sufficiently large $\nu \in \mathbb{N}$ and some constant C

$$\|F_{A^\nu}(r^\nu,\phi^\nu)\|_{L^\infty(\Sigma)} \;\leq\; C(R^\nu)^2 \;=\; 4C\Delta^2(r^\nu)^{-2}$$

in contradiction to $(r^\nu)^2\,\|F_{A^\nu}(r^\nu,\phi^\nu)\|_{L^\infty(\Sigma)} \to \infty$.

Step 3: *Proof of (i)*
Fix a connection Ξ as assumed and consider a point (r,ϕ) with $\mathcal{E}(2r)\leq\varepsilon$. Here we first choose $\varepsilon>0$ as in step 2. The L^∞-bound from step 2 can be put into the estimate of lemma 2.3 to find another constant C such that on $B_{\frac{1}{2}r}(r,\phi)$

$$\Delta\big\|F_\Xi\big\|^2_{L^2(\Sigma)} \;\leq\; Cr^{-2}\big\|F_\Xi\big\|^2_{L^2(\Sigma)}.$$

In the boundary case this lemma also provides

$$-\tfrac{\partial}{\partial t}\big|_{t=0}\big\|F_\Xi\big\|^2_{L^2(\Sigma)} \;\leq\; C\big(\big\|F_\Xi\big\|^2_{L^2(\Sigma)}+\big\|F_\Xi\big\|^3_{L^2(\Sigma)}\big).$$

Now we can choose a smaller $\varepsilon>0$ such that $\varepsilon\leq\frac{1}{3}\mu C^{-2}$ with the $\mu>0$ from proposition 2.5. Then we obtain the following mean value inequality for the function $\|F_\Xi\|^2_{L^2(\Sigma)}$ with another constant C',

$$\big\|F_\Xi(r,\phi)\big\|^2_{L^2(\Sigma)} \;\leq\; C'r^{-2}\int_{B_{\frac{1}{2}r}(r,\phi)}\big\|F_\Xi\big\|^2_{L^2(\Sigma)} \;\leq\; 2C'r^{-2}\mathcal{E}(2r).$$

Step 4: *Proof of (ii),(ii')*
It suffices to prove the estimates for $r\leq\bar{r}_0$ with some fixed $\bar{r}_0>0$, since then in case $\bar{r}_0<r\leq\frac{1}{2}r_0$ (and similarly in the boundary case)

$$\|F_\Xi(r,\phi)\|^2_{L^\infty(\Sigma)} \;\leq\; C(\bar{r}_0)^{-4}\mathcal{E}(2\bar{r}_0) \;\leq\; C\Big(\frac{r_0}{2\bar{r}_0}\Big)^4 r^{-4}\mathcal{E}(2r).$$

First, let $\bar{r}_0>0$ be the minimum of the injectivity radius on Σ for the metrics $g_{s,t}$. Then we choose $\bar{r}_0>0$ even smaller such that the pullback of all these metrics under normal coordinates on a ball of radius $\bar{r}_0$ is $\mathcal{C}^1$-close to the Euclidean metric on $\mathbb{R}^2$. Thus we will be able to work with uniform constants C and $\mu>0$ in proposition 2.2.

In the interior case we consider a connection Ξ as assumed and any point $(r,\phi,z)\in B^*_{\bar{r}_0}\times\Sigma$. The normal coordinates centered at this point give a coordinate chart on $B_{\frac{1}{2}r}(0)\subset\mathbb{R}^4$. From lemma 2.3 we have a uniform constant C such that on $B_{\frac{1}{2}r}(r,\phi,z)\subset B^*\times\Sigma$

$$\Delta\big|F_\Xi\big|^2 \;\leq\; C\big|F_\Xi\big|^2+8\big|F_\Xi\big|^3.$$

Now let $0<\varepsilon\leq\frac{\mu}{65}$, then proposition 2.2 applies to the pullback of the function $|F_\Xi|^2$ on the coordinate chart $B_{\frac{1}{2}r}(0)$ and asserts that

$$\big|F_\Xi(r,\phi,z)\big|^2 \;\leq\; C\big(1+r^{-4}\big)\int_{B_{\frac{1}{2}r}(r,\phi,z)}\big|F_\Xi\big|^2 \;\leq\; Cr^{-4}\mathcal{E}(2r).$$

Here C denotes any finite constant and we have used $1\leq(r_0)^4r^{-4}$.

In the boundary case on $D^*\times\Sigma$ we use the same mean value inequality on the ball $B_\rho(r,\phi,z)\subset D^*\times\Sigma$ of radius $\rho=\frac{1}{2}r\sin\phi$ for any $(r,\phi,z)\in D^*_{\bar{r}_0}\times\Sigma$

and $0 < \phi < \pi$. The normal coordinates centered at (r,ϕ,z) give a coordinate chart on the full ball $B_\rho(0)\subset\mathbb{R}^4$. With the same estimate on $\Delta|F_\Xi|^2$ and the same $\varepsilon>0$ as above we then apply proposition 2.2 to obtain

$$\bigl|F_\Xi(r,\phi,z)\bigr|^2 \;\le\; C\bigl(1+\rho^{-4}\bigr)\int_{B_\rho(r,\phi,z)}\bigl|F_\Xi\bigr|^2 \;\le\; C(r\sin\phi)^{-4}\mathcal{E}(2r).$$

Again, C denotes any finite constant, and $1\le(r_0)^4(r\sin\phi)^{-4}$. $\square$

Proof of lemma 5.1 and remark 5.2: We will use C and C' to denote all finite constants. These might depend on the connection Ξ.

$\underline{(i)\Rightarrow(ii):}$ Since $\mathcal{E}(r_0)<\infty$ we must have $\mathcal{E}(r)\to 0$ as $r\to 0$. So we find $\bar r>0$ such that for all $0<r\le\bar r$ we obtain from lemma 5.4

$$\sup_\phi \|F_\Xi(r,\phi)\|_{L^2(\Sigma)} \;\le\; r^{-1}\sqrt{C\,\mathcal{E}(2r)} \;\le\; C' r^{\beta-1}.$$

For $\bar r<r\le r_0$ we have with a constant C' depending on $\bar r$ or r_0

$$\sup_\phi \|F_\Xi(r,\phi)\|_{L^2(\Sigma)} \;\le\; \sup_{r\in[\bar r,r_0]}\sup_\phi \|F_\Xi(r,\phi)\|_{L^2(\Sigma)} \;=\; C \;\le\; C' r^{\beta-1}.$$

$\underline{(ii)\Rightarrow(i):}$ Without using (19) or (17) we can simply calculate for all $\rho\le\frac12 r_0$

$$\mathcal{E}(\rho) \;=\; \tfrac12\int_0^\rho\int \|F_\Xi(r,\phi)\|^2_{L^2(\Sigma)}\, r\,\mathrm{d}\phi\,\mathrm{d}r \;\le\; \pi\int_0^\rho C^2 r^{2\beta-1}\,\mathrm{d}r \;\le\; C'\rho^{2\beta}.$$

This already implies $\mathcal{E}(r_0)<\infty$. Then for $\frac12 r_0<\rho\le r_0$ we have

$$\mathcal{E}(\rho) \;\le\; \mathcal{E}(r_0)\,(\tfrac12 r_0)^{-2\beta}\rho^{2\beta} \;=\; C\rho^{2\beta}.$$

$\underline{(i)\Rightarrow(iv),(iv'):}$ Since Ξ is smooth away from $\{0\}\times\Sigma$ it suffices to establish the estimates for all $0<r\le\bar r$. We pick $\bar r>0$ such that the assumptions of lemma 5.4 are satisfied, in particular $\mathcal{E}(2\bar r)\le\varepsilon$. Then in the boundary case and the interior case respectively the lemma asserts

$$\begin{aligned}
\|F_\Xi(r,\phi)\|_{L^\infty(\Sigma)} &\;\le\; (r\sin\phi)^{-2}\sqrt{C\,\mathcal{E}(2r)} \;\le\; C' r^{\beta-2}(\sin\phi)^{-2} \qquad &&\forall\phi\in(0,\pi),\\
\|F_\Xi(r,\phi)\|_{L^\infty(\Sigma)} &\;\le\; r^{-2}\sqrt{C\,\mathcal{E}(2r)} \;\le\; C' r^{\beta-2} &&\forall\phi\in[0,2\pi].
\end{aligned}$$

$\underline{(i)\Rightarrow(iii):}$ This works the same for $D^*\times\Sigma$ and $B^*\times\Sigma$, so we only consider the first case. (In the second case, the $\sin\phi$-factor can be dropped.) We already know that (i) implies (ii) and (iv). Then just working with these two assumptions, we can interpolate for all $p>2$

$$\begin{aligned}
\bigl\|F_\Xi\bigr\|^p_{L^p(D^*\times\Sigma)} &\;=\; \lim_{\delta\to 0}\int_{(D_{r_0}\setminus D_\delta)\times\Sigma}\bigl|F_\Xi\bigr|^p\\
&\;=\; \lim_{\delta\to 0}\int_\delta^{r_0}\int_0^\pi \bigl\|F_\Xi(r,\phi)\bigr\|^{p-2}_{L^\infty(\Sigma)}\bigl\|F_\Xi(r,\phi)\bigr\|^2_{L^2(\Sigma)}\, r\,\mathrm{d}\phi\,\mathrm{d}r\\
&\;\le\; \lim_{\delta\to 0} C\int_\delta^{r_0}\int_0^\pi r^{(\beta-2)(p-2)+2(\beta-1)}(\sin\phi)^{-2(p-2)}\, r\,\mathrm{d}\phi\,\mathrm{d}r\\
&\;\le\; \lim_{\delta\to 0} 2C\int_0^{\frac{\pi}{2}}\bigl(\tfrac{2}{\pi}\phi\bigr)^{-2(p-2)}\,\mathrm{d}\phi\int_\delta^{r_0} r^{(\beta-2)p+3}\,\mathrm{d}r.
\end{aligned}$$

Here we use $\sin\phi \geq \frac{2}{\pi}\phi$ for $\phi \in [0, \frac{\pi}{2}]$. The ϕ-integral is finite for $p < \frac{5}{2}$, and the r-integral converges to a finite value if $(\beta - 2)p > -4$. So if $\beta \geq 2$, then we just need $2 < p < \frac{5}{2}$, and if $\beta < 2$, then we need in addition $p < \frac{4}{2-\beta}$.

$\underline{(iii) \Rightarrow (i):}$ This is the same calculation for both $D^* \times \Sigma$ and $B^* \times \Sigma$, and it works without the assumption (19) or (17). In the first case for all $r \leq r_0$,

$$\begin{aligned}\mathcal{E}(r) \;&=\; \lim_{\delta\to 0} \tfrac{1}{2} \int_{(D_r \setminus D_\delta)\times\Sigma} |F_\Xi|^2 \\ &\leq\; \tfrac{1}{2}\mathrm{Vol}(D_r \times \Sigma)^{1-\frac{2}{p}} \;\lim_{\delta\to 0}\Bigl(\int_{(D_r \setminus D_\delta)\times\Sigma} |F_\Xi|^p\Bigr)^{\frac{2}{p}} \;\leq\; C' r^{2(1-\frac{2}{p})}.\end{aligned}$$

□

Proof of theorem 5.3: We will give the full proof in the boundary case (ii) and point out where it differs (mostly simplifies) in the interior case (i).

Given a connection $\Xi \in \mathcal{A}(D^* \times \Sigma)$ as assumed we first put it into the special gauge $\Xi = A + R\mathrm{d}r$ with $A : D^* \to \mathcal{A}(\Sigma)$ and $R : D^* \to \mathcal{C}^\infty(\Sigma, \mathfrak{g})$ such that $R|_{\phi=\frac{\pi}{2}} \equiv 0$ (and $\Phi \equiv 0$). This is achieved by a gauge transformation $u \in \mathcal{G}(D^* \times \Sigma)$ that is determined as follows: For every $z \in \Sigma$ first solve $\partial_r u = -Ru$ with initial value $u(r_0, \frac{\pi}{2}, z) = \mathbb{1}$, to determine $u(\cdot, \frac{\pi}{2}, z)$, then for each $r \in (0, r_0]$ use this as initial value and solve $\partial_\phi u = -\Phi u$ to obtain $u(r, \phi, z)$ for all $\phi \in [0, \pi]$. That way the gauge is fixed up to a gauge transformation on Σ, i.e. independent of $(r, \phi) \in D^*$. (In case (i) this construction does in general not yield $u(r, 0, z) = u(r, 2\pi, z)$ and hence define a gauge transformation on $B^* \times \Sigma$. Thus the existence of this gauge is an assumption in the theorem. Given this gauge, one then only needs to solve $\partial_r u = -Ru$ at $\phi = \frac{\pi}{2}$.) In this gauge and splitting, the norm of the curvature is

$$|F_\Xi|^2 = |F_A|^2 + |\partial_r A - \mathrm{d}_A R|^2 + r^{-2}|\partial_\phi R|^2 + r^{-2}|\partial_\phi A|^2.$$

In particular, note that

$$|\partial_\phi \Xi|^2 = |\partial_\phi R|^2 + |\partial_\phi A|^2 \leq r^2 |F_\Xi|^2, \qquad |\partial_r \Xi|^2_{\phi=\frac{\pi}{2}} = |\partial_r A|^2_{\phi=\frac{\pi}{2}} \leq |F_\Xi|^2_{\phi=\frac{\pi}{2}}.$$

Next, we can combine the assumptions (ii) and (iv) as in lemma 5.1 to obtain for any $q > 2$ (in case (i) even without the $\sin\phi$-term)

$$\|F_\Xi(r,\phi)\|^q_{L^q(\Sigma)} \;\leq\; \|F_\Xi(r,\phi)\|^{q-2}_{L^\infty(\Sigma)} \|F_\Xi(r,\phi)\|^2_{L^2(\Sigma)} \;\leq\; C\, r^{2-(2-\beta)q} (\sin\phi)^{4-2q}.$$

By integrating this over D^* we recover (iii) of lemma 5.1: If $2 < p < \frac{5}{2}$ (in case (i) we only need $p > 2$) and $\frac{1}{p} > \frac{2-\beta}{4}$ then

$$\|F_\Xi\|^p_{L^p(D_\rho \times \Sigma)} \;\leq\; C\, \rho^{4-(2-\beta)p} \;\underset{\rho\to 0}{\longrightarrow}\; 0. \tag{23}$$

Moreover, for any $q > 2$ we can read off for all $0 < r \leq r_0$ and $0 < \phi < \pi$

$$\begin{aligned}\|\partial_\phi \Xi(r,\phi)\|_{L^q(\Sigma)} &\leq C\, r^{\frac{2}{q}+\beta-1} (\sin\phi)^{\frac{4}{q}-2}, \\ \|\partial_r \Xi(r, \tfrac{\pi}{2})\|_{L^q(\Sigma)} &\leq C\, r^{\frac{2}{q}+\beta-2}.\end{aligned} \tag{24}$$

Integrating the second estimate shows that for $\frac{1}{p} > \frac{1-\beta}{2}$ there exists a limit $\Xi(r, \frac{\pi}{2}) = A(r, \frac{\pi}{2}) \to A_0 \in \mathcal{A}^{0,p}(\Sigma)$ as $r \to 0$. The first estimate then implies $\Xi(r, \cdot) \to A_0$ in $\mathcal{C}^0([0, \pi], \mathcal{A}^{0,p}(\Sigma))$. This motivates the following construction:

Fix a smooth cutoff function $h : [0, \infty) \to [0, 1]$ with $h|_{[0,\varepsilon]} \equiv 0$ and $h|_{[1-\varepsilon,\infty)} \equiv 1$ for some $\varepsilon > 0$ and such that $|h'| \leq 2$. Now for every $0 < \rho \leq \frac{1}{2}r_0$ we set $A_\rho := \Xi(\rho, \frac{\pi}{2}) \in \mathcal{A}(\Sigma)$ and define $\Xi^\rho \in \mathcal{A}(D \times \Sigma)$ by

$$\Xi^\rho(r, \phi) \;:=\; A_\rho + h(\tfrac{r}{\rho})\big(\Xi(r, \phi) - A_\rho\big).$$

Note that $\Xi^\rho|_{D \setminus D_\rho} = \Xi|_{D \setminus D_\rho}$. We will find gauges for some sequence Ξ^{ρ_i}, $\rho_i \to 0$ such that these connections converge $W^{1,p}$-weakly. The limit will then be the extended connection $\tilde{\Xi} \in \mathcal{A}^{1,p}(D \times \Sigma)$, and the gauge transformations will converge on $D^* \times \Sigma$ to $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}$ such that $u^*\Xi = \tilde{\Xi}|_{D^* \times \Sigma}$. This weak limit will be a consequence of Uhlenbeck's weak compactness theorem, so we have to control the curvatures

$$\begin{aligned}
F_{\Xi^\rho} =\; & \mathrm{d}A_\rho \;+\; h(\tfrac{r}{\rho})\big(\mathrm{d}\Xi - \mathrm{d}A_\rho\big) \;-\; \tfrac{1}{\rho}h'(\tfrac{r}{\rho})\big(\Xi - A_\rho\big) \wedge \mathrm{d}r \\
& + \tfrac{1}{2}[A_\rho \wedge A_\rho] \;+\; \tfrac{1}{2}h(\tfrac{r}{\rho})^2\big[(\Xi - A_\rho) \wedge (\Xi - A_\rho)\big] \;+\; h(\tfrac{r}{\rho})\big[A_\rho \wedge (\Xi - A_\rho)\big] \\
=\; & \big(1 - h(\tfrac{r}{\rho})\big)F_{A_\rho} \;+\; h(\tfrac{r}{\rho})F_\Xi \;-\; \tfrac{1}{\rho}h'(\tfrac{r}{\rho})\big(\Xi - A_\rho\big) \wedge \mathrm{d}r \\
& + \tfrac{1}{2}\big(h(\tfrac{r}{\rho})^2 - h(\tfrac{r}{\rho})\big)\big[(\Xi - A_\rho) \wedge (\Xi - A_\rho)\big].
\end{aligned}$$

From (23) we know that $F_\Xi \in L^p(D \times \Sigma)$. Now we shall see that $F_{\Xi^\rho} \to F_\Xi$ in $L^p(D \times \Sigma)$ as $\rho \to 0$:

$$\begin{aligned}
\big\|F_{\Xi^\rho} - F_\Xi\big\|_{L^p(D\times\Sigma)} \leq\; & \big\|F_{A_\rho}\big\|_{L^p(D_\rho\times\Sigma)} + \big\|F_\Xi\big\|_{L^p(D_\rho\times\Sigma)} \\
& + \tfrac{2}{\rho}\big\|\Xi - A_\rho\big\|_{L^p(D_\rho\times\Sigma)} + \big\|\Xi - A_\rho\big\|^2_{L^{2p}(D_\rho\times\Sigma)}.
\end{aligned}$$

The second term on the right hand side converges to zero by (23). For the first term we use (24) and recall that $p > 2$ such that $\frac{1}{p} > \frac{2-\beta}{4}$, so

$$\big\|F_{A_\rho}\big\|^p_{L^p(D_\rho\times\Sigma)} \;=\; \int_{D_\rho} \big\|F_\Xi(\rho, \tfrac{\pi}{2})\big\|^p_{L^p(\Sigma)} \;\leq\; \tfrac{1}{2}\pi C\, r^{4-(2-\beta)p} \underset{\rho\to 0}{\longrightarrow} 0.$$

To control the other two terms we first calculate for general $q > 2$, assuming $q \neq 4$, $\frac{2}{q} + \beta \neq 1$, and denoting all constants by C

$$\begin{aligned}
& \big\|\Xi - A_\rho\big\|^q_{L^q(D_\rho\times\Sigma)} \\
& = \int_{D_\rho} \bigg\| \int_\rho^r \partial_r \Xi(t, \tfrac{\pi}{2})\,\mathrm{d}t + \int_{\frac{\pi}{2}}^\phi \partial_\phi \Xi(r, \theta)\,\mathrm{d}\theta \bigg\|^q_{L^q(\Sigma)} \qquad (25) \\
& \leq C \int_0^\rho \int_0^{\frac{\pi}{2}} \bigg(\int_r^\rho t^{\frac{2}{q}+\beta-2}\,\mathrm{d}t \;+\; \int_\phi^{\frac{\pi}{2}} r^{\frac{2}{q}+\beta-1} (\sin\theta)^{\frac{4}{q}-2}\,\mathrm{d}\theta \bigg)^q r\,\mathrm{d}\phi\,\mathrm{d}r \\
& \leq C \int_0^\rho \bigg(r\rho^{2-(1-\beta)q} + r^{3-(1-\beta)q} + r^{3-(1-\beta)q} \int_0^{\frac{\pi}{2}} \big(1 - \big(\tfrac{2}{\pi}\phi\big)^{\frac{4}{q}-1}\big)^q\,\mathrm{d}\phi \bigg)\,\mathrm{d}r \\
& \leq C\rho^{4-(1-\beta)q}.
\end{aligned}$$

Here we have used the fact that $\sin\theta \geq \frac{2}{\pi}\theta$ for $\theta \in [0, \frac{\pi}{2}]$. The ϕ-integral then gives a finite value for $q < 5$ and the r-integral converges for $\frac{1}{q} > \frac{1-\beta}{4}$. For $\frac{2}{q} + \beta = 1$ we have to deal differently with the t-integral in (25), but still

$$\int_0^\rho \left(\int_r^\rho t^{-1}\,\mathrm{d}t \right)^q r\,\mathrm{d}r \;=\; \int_0^\rho r\,\ln\!\left(\tfrac{\rho}{r}\right)^q \mathrm{d}r \;=\; \int_1^\infty \rho^2 e^{-2y} y^q\,\mathrm{d}y \;=\; C\rho^2.$$

So (25) holds for $2 < q < 5$ if $q \neq 4$ and $\frac{1}{q} > \frac{1-\beta}{4}$. These conditions are all satisfied for $q = p$ since $\frac{2-\beta}{4} > \frac{1-\beta}{2}$. So (25) implies

$$\tfrac{2}{\rho}\big\|\Xi - A_\rho\big\|_{L^p(D_\rho\times\Sigma)} \;\leq\; C\rho^{\frac{4}{p}+\beta-2} \underset{\rho\to 0}{\longrightarrow} 0.$$

Finally, we can choose $q = 2p$ in (25) since then $4 < q < 5$ and $\frac{1}{q} > \frac{2-\beta}{8} > \frac{1-\beta}{4}$. If we also note that $\frac{2}{p} > \frac{2-\beta}{2} > 1 - \beta$, then this gives

$$\big\|\Xi - A_\rho\big\|_{L^{2p}(D_\rho\times\Sigma)} \;\leq\; C\rho^{\frac{2}{p}+\beta-1} \underset{\rho\to 0}{\longrightarrow} 0.$$

Thus we have checked that $\|F_{\Xi^\rho} - F_\Xi\|_{L^p(D\times\Sigma)} \to 0$ as $\rho \to 0$, and hence $\|F_{\Xi^\rho}\|_{L^p(D\times\Sigma)}$ must be bounded for $\rho \in (0, \frac{1}{2}r_0]$. In order to apply Uhlenbeck's weak compactness theorem ([U2, Thm 1.5] or [W1, Thm A]), we choose a closed subset $D_{\frac{1}{2}r_0} \subset U \subset \mathrm{int}(D)$ with smooth boundary, and we denote $U^* = U \setminus \{0\}$. Then for some sequence $\rho_i \to 0$ there exist gauge transformations $u_i \in \mathcal{G}^{2,p}(U\times\Sigma)$ such that the gauge transformed connections $u_i^*\Xi^{\rho_i}$ converge $W^{1,p}$-weakly to some $\tilde{\Xi} \in \mathcal{A}^{1,p}(U\times\Sigma)$. On every compact subset $K \subset U^*\times\Sigma$ we have $\|\Xi^{\rho_i} - \Xi\|_{W^{1,p}(K)} \to 0$. In particular both $\|\Xi^{\rho_i}\|_{W^{1,p}(K)}$ and $\|u_i^*\Xi^{\rho_i}\|_{W^{1,p}(K)}$ are bounded and thus $\|u_i^{-1}\mathrm{d}u_i\|_{W^{1,p}(K)}$ is bounded. Hence for some further subsequence, $u_i|_{U^*\times\Sigma}$ converges to some $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(U^*\times\Sigma)$ in the $\mathcal{C}^0$-topology and in the weak $W^{2,p}$-topology on every compact subset (see e.g. [W1, Lemma A.8]). Furthermore, $u^*\Xi|_{U^*\times\Sigma} = \tilde{\Xi}|_{U^*\times\Sigma}$ since on every compact subset both are the weak $W^{1,p}$-limit of $u_i^*\Xi^{\rho_i}$.

On $(D\setminus U)\times\Sigma$ we can now choose an extension of u and define $\tilde{\Xi} = u^*\Xi$ to obtain the claimed gauge transformation $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(D^*\times\Sigma)$ and extension $\tilde{\Xi} \in \mathcal{A}^{1,p}(D\times\Sigma)$ with $u^*\Xi = \tilde{\Xi}|_{D^*\times\Sigma}$ The interior case (i) is proven exactly the same way. Just the estimates are simplified due to the absence of the $\sin\phi$-term.

Furthermore, if Ξ is anti-self-dual, then in both cases we also know that $\tilde{\Xi}$ is anti-self-dual since $\|F_{\tilde{\Xi}} + *F_{\tilde{\Xi}}\|_{L^p(D\times\Sigma)} = \|F_{u^*\Xi} + *F_{u^*\Xi}\|_{L^p(D\times\Sigma)} = 0$. Finally, suppose that Ξ has Lagrangian boundary values $\Xi|_{(s,0)\times\Sigma} \in \mathcal{L}_Y$ for all $0 < |s| \leq r_0$. Since $\mathcal{L}_Y$ is gauge invariant and $\Xi^\rho|_{\{r\geq\rho\}} = \Xi|_{\{r\geq\rho\}}$ we thus know for every $0 < |s| \leq r_0$ that $u_i^*\Xi^{\rho_i}|_{(s,0)\times\Sigma} \in \mathcal{L}_Y$ for all sufficiently large $i \in \mathbb{N}$. Moreover, $u_i^*\Xi^{\rho_i}$ is bounded in $W^{1,p}(D\times\Sigma)$, and the embedding $W^{1,p}(D\times\Sigma) \hookrightarrow \mathcal{C}^0(D, L^p(\Sigma))$ is compact (see [W3, Lemma 2.5]). So some subsequence of $u_i^*\Xi^{\rho_i}|_{(s,0)\times\Sigma}$ converges in $\mathcal{A}^{0,p}(\Sigma)$ for all $-r_0 \leq s \leq r_0$. Since $\mathcal{L}_Y \subset \mathcal{A}^{0,p}(\Sigma)$ is closed this implies $\tilde{\Xi}|_{(s,0)\times\Sigma} \in \mathcal{L}_Y$ for all $0 < |s| \leq r_0$. This also holds at $s = 0$ since $\tilde{\Xi}|_{(s,0)\times\Sigma} \in \mathcal{A}^{0,p}(\Sigma)$ is a continuous path for $s \in [-r_0, r_0]$ by the embedding $W^{1,p}(D\times\Sigma) \hookrightarrow \mathcal{C}^0(D, L^p(\Sigma))$. □

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